Continuous-Time Fourier Transform

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- Introduction
- Fourier Integral
- Fourier Transform
- Properties of Fourier Transform
- Convolution
- Parseval's Theorem

Continuous-Time Fourier Transform

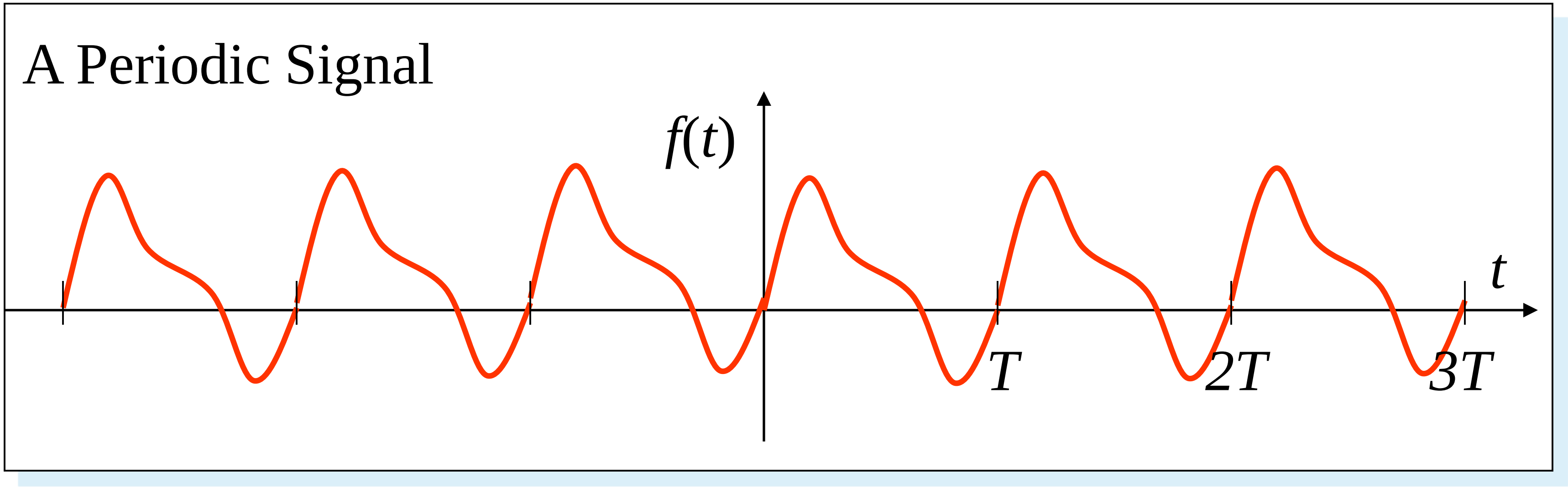
Introduction

The Topic

	Continuous Time	Discrete Time
Periodic	Fourier Series	Discrete Fourier Transform
Aperiodic	Continuous Fourier Transform	Fourier Transform

Review of Fourier Series

- Deal with continuous-time periodic signals.
- Discrete frequency spectra.



Two Forms for Fourier Series

Sinusoidal Form

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{2\pi nt}{T} + \sum_{n=1}^{\infty} b_n \sin \frac{2\pi nt}{T}$$

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} f(t)dt$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos n\omega_0 t dt$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin n\omega_0 t dt$$

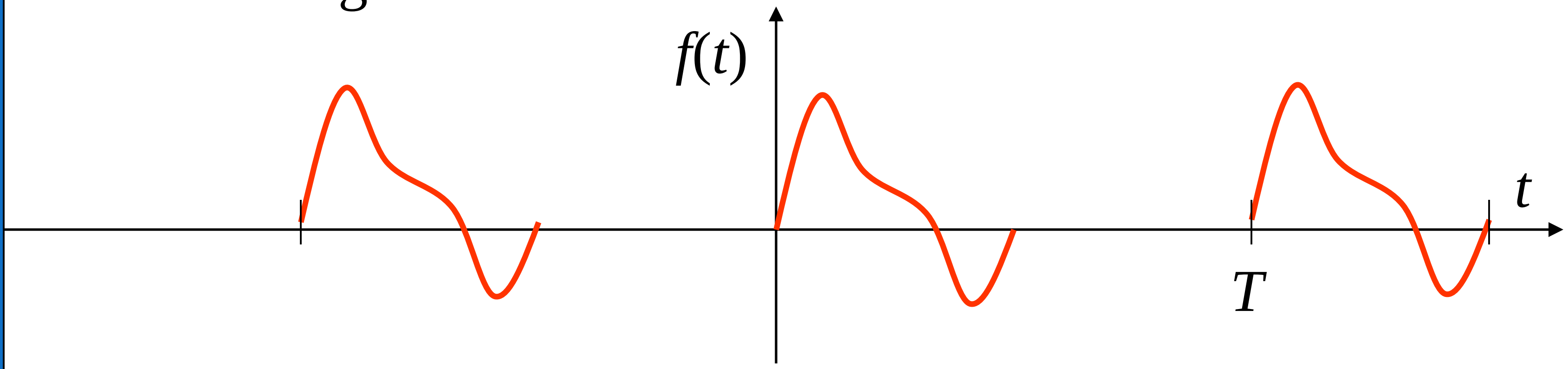
Complex Form:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jn\omega_0 t} dt$$

How to Deal with Aperiodic Signal?

A Periodic Signal



If $T \to \infty$, what happens?

Continuous-Time Fourier Transform

Fourier Integral

Fourier Integral

$$f_T(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} \qquad c_n = \frac{1}{T}\int_{-T/2}^{T/2} f_T(t) e^{-jn\omega_0 t}\,dt$$

$$= \sum_{n=-\infty}^{\infty}\left[\frac{1}{T}\int_{-T/2}^{T/2} f_T(\tau) e^{-jn\omega_0 \tau}\,d\tau\right] e^{jn\omega_0 t}$$

$$\omega_0 = \frac{2\pi}{T} \Rightarrow \frac{1}{T} = \frac{\omega_0}{2\pi}$$

$$= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\left[\int_{-T/2}^{T/2} f_T(\tau) e^{-jn\omega_0 \tau}\,d\tau\right] \omega_0 e^{jn\omega_0 t}$$

$$\text{Let } \Delta\omega = \omega_0 = \frac{2\pi}{T}$$

$$= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\left[\int_{-T/2}^{T/2} f_T(\tau) e^{-jn\omega_0 \tau}\,d\tau\right] e^{jn\omega_0 t}\Delta\omega$$

$$T \to \infty \Rightarrow d\omega = \Delta\omega \approx 0$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(\tau) e^{-j\omega\tau}\,d\tau\right] e^{j\omega t}\,d\omega$$

Fourier Integral

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\underbrace{\left[\int_{-\infty}^{\infty}f(\tau)e^{-j\omega\tau}d\tau\right]}_{F(j\omega)}e^{j\omega t}d\omega$$

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(j\omega)e^{j\omega t}d\omega$$ Synthesis

$$F(j\omega)=\int_{-\infty}^{\infty}f(t)e^{-j\omega t}dt$$ Analysis

Fourier Series vs. Fourier Integral

Fourier Series:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

Period Function

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f_T(t) e^{-jn\omega_0 t} dt$$

Discrete Spectra

Fourier Integral:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega) e^{j\omega t} d\omega$$

Non-Period Function

$$F(j\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

Continuous Spectra

Continuous-Time Fourier Transform

Fourier Transform

Fourier Transform Pair

Inverse Fourier Transform:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}d\omega$$

Synthesis

Fourier Transform:

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

Analysis

Existence of the Fourier Transform

Sufficient Condition:

$f(t)$ is absolutely integrable, i.e.,

$$\int_{-\infty}^{\infty} |f(t)| dt < \infty$$

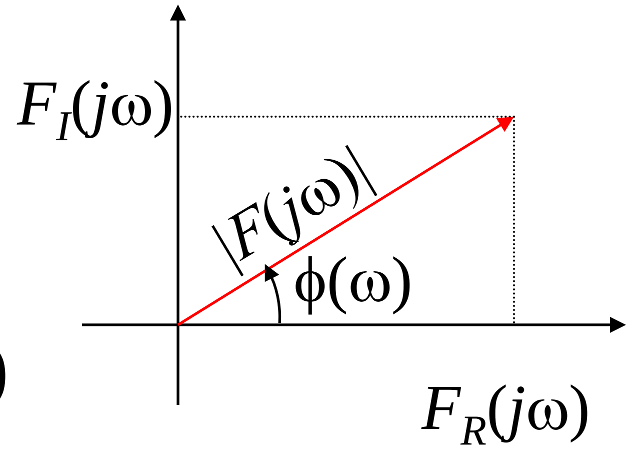
Continuous Spectra

$$F(j\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

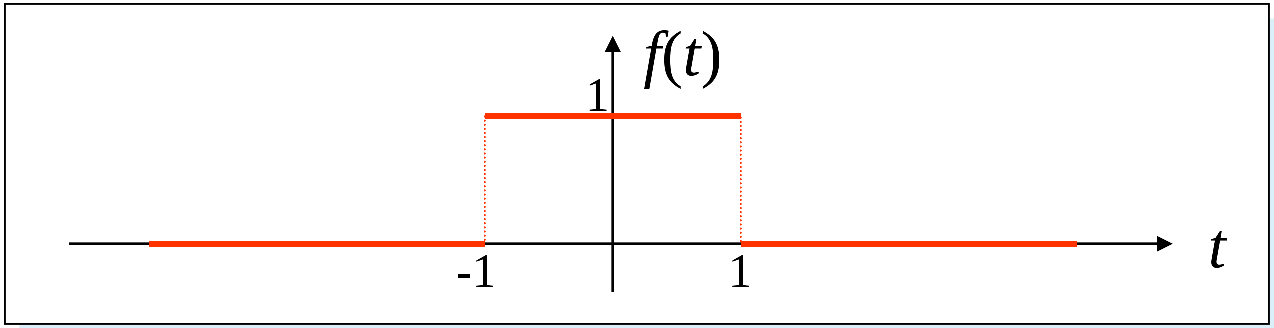
$$F(j\omega) = F_R(j\omega) + jF_I(j\omega)$$

$$= \underbrace{|F(j\omega)|}_{\text{Magnitude}} e^{\overbrace{j\phi(\omega)}^{}}$$

Phase



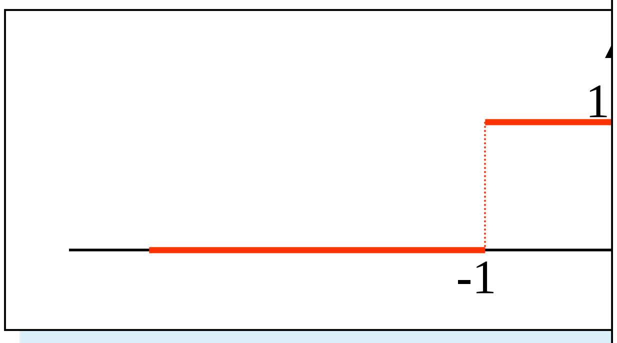
Example

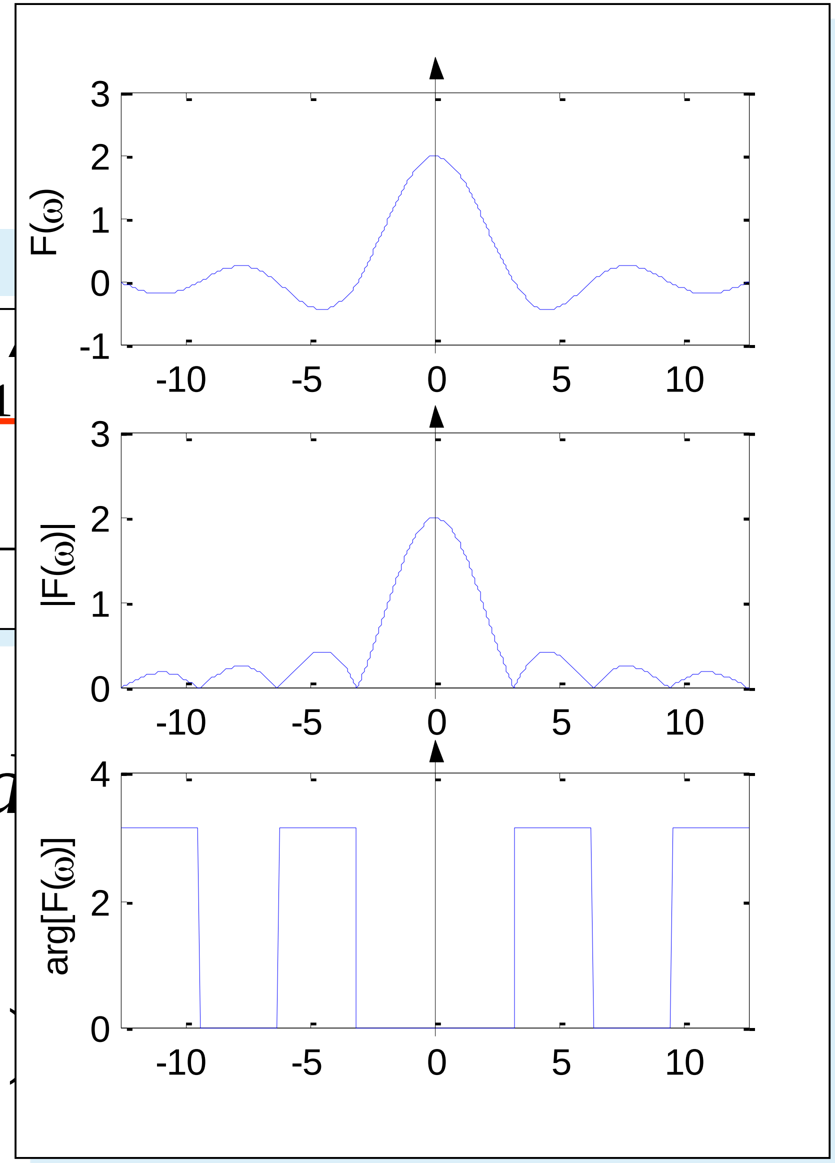


$$F(j\omega)=\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt=\int_{-1}^{1} e^{-j\omega t}dt=\frac{1}{-j\omega}e^{-j\omega t}\Big|_{-1}^{1}$$

$$=\frac{j}{\omega}(e^{-j\omega}-e^{j\omega})=\frac{2\sin\omega}{\omega}=2\sin c\,2\pi f$$

Example



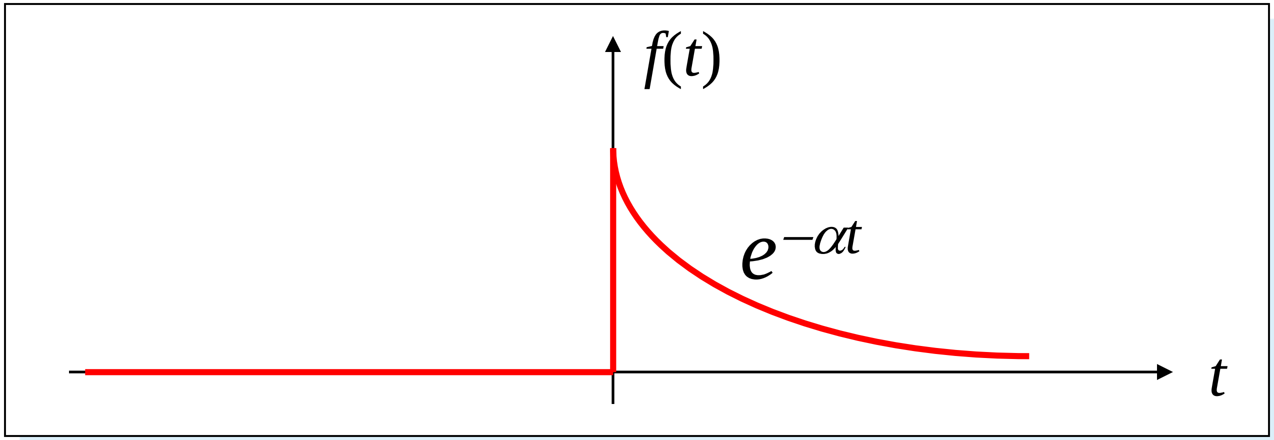


$$F(j\omega)=\int_{-\infty}^{\infty} f(t)e^{-j\omega t}d$$

$t\Big|_{-1}^{1}$

$$=\frac{j}{\omega}(e^{-j\omega}-e^{j\omega}$$

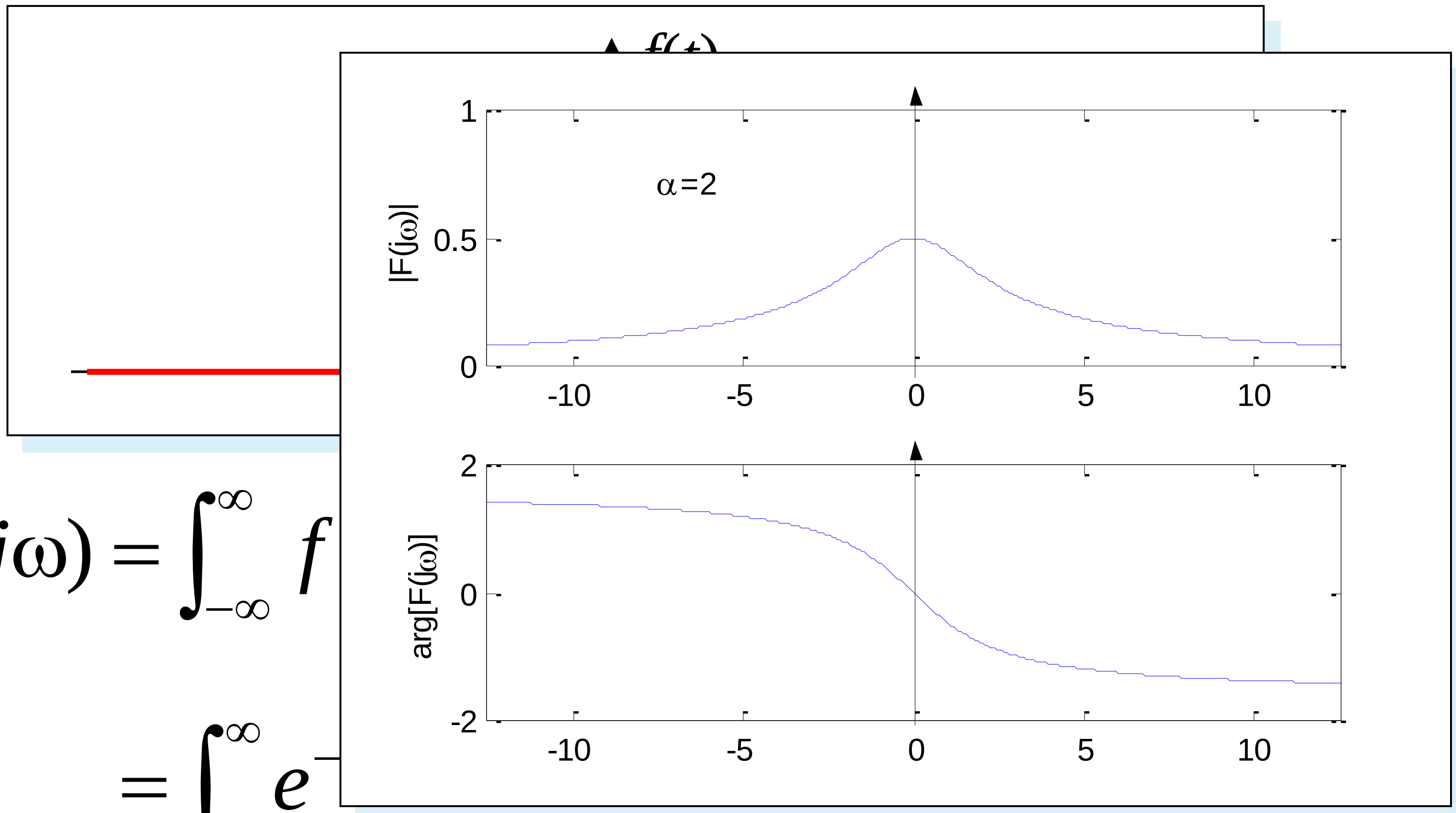
Example



$$F(j\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \;\; = \int_{0}^{\infty} e^{-\alpha t} e^{-j\omega t} dt$$

$$= \int_{0}^{\infty} e^{-(\alpha + j\omega)t} dt \;\; = \frac{1}{\alpha + j\omega}$$

Example



$$F(j\omega) = \int_{-\infty}^{\infty} f$$

$$= \int_{0}^{\infty} e^{-}$$

$$\alpha + j\omega$$

Continuous-Time Fourier Transform

Properties of
Fourier Transform

Notation

$$\mathcal{F}[f(t)] = F(j\omega)$$

$$\mathcal{F}^{-1}[F(j\omega)] = f(t)$$

$$\Rightarrow \quad f(t) \xleftrightarrow{\mathcal{F}} F(j\omega)$$

Linearity

$$a_1 f_1(t) + a_2 f_2(t) \xleftrightarrow{\mathcal{F}} a_1 F_1(j\omega) + a_2 F_2(j\omega)$$

Proved by yourselves

Time Scaling

$$f(at) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} F\left(j \frac{\omega}{a} \right)$$

Proved by yourselves

Time Reversal

$$f(-t) \overset{\mathcal{F}}{\longleftrightarrow} F(-j\omega)$$

Pf)

$$\mathcal{F}[f(-t)] = \int_{-\infty}^{\infty} f(-t)e^{-j\omega t}dt = \int_{t=-\infty}^{t=\infty} f(-t)e^{-j\omega t}dt$$

$$= \int_{-t=-\infty}^{-t=\infty} f(t)e^{j\omega t}d(-t) = \int_{-t=-\infty}^{-t=\infty} f(t)e^{j\omega t}d(-t)$$

$$= -\int_{t=\infty}^{t=-\infty} f(t)e^{j\omega t}dt = \int_{t=-\infty}^{t=\infty} f(t)e^{j\omega t}dt$$

$$= \int_{-\infty}^{\infty} f(t)e^{j\omega t}dt = F(-j\omega)$$

Time Shifting

$$f(t-t_0) \overset{\mathcal{F}}{\longleftrightarrow} F(j\omega)e^{-j\omega t_0}$$

Pf)

$$\mathcal{F}[f(t-t_0)] = \int_{-\infty}^{\infty} f(t-t_0)e^{-j\omega t}dt = \int_{t=-\infty}^{t=\infty} f(t-t_0)e^{-j\omega t}dt$$

$$= \int_{t+t_0=-\infty}^{t+t_0=\infty} f(t)e^{-j\omega(t+t_0)}d(t+t_0)$$

$$= e^{-j\omega t_0}\int_{t=-\infty}^{t=\infty} f(t)e^{-j\omega t}dt$$

$$= e^{-j\omega t_0}\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt = F(j\omega)e^{-j\omega_0 t}$$

Frequency Shifting (Modulation)

$$f(t)e^{j\omega_0} \xleftrightarrow{\mathcal{F}} F\left[j(\omega-\omega_0)\right]$$

Pf)

$$\mathcal{F}[f(t)e^{j\omega_0 t}] = \int_{-\infty}^{\infty} f(t)e^{j\omega_0 t}e^{-j\omega t}dt$$

$$= \int_{-\infty}^{\infty} f(t)e^{-j(\omega-\omega_0)t}dt$$

$$= F\left[j(\omega-\omega_0)\right]$$

Symmetry Property

$$\mathcal{F}[F(jt)] = 2\pi f(-\omega)$$

Pf)

$$2\pi f(t) = \int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}d\omega$$

$$2\pi f(-t) = \int_{-\infty}^{\infty} F(j\omega)e^{-j\omega t}d\omega$$

Interchange symbols ω and t

$$2\pi f(-\omega) = \int_{-\infty}^{\infty} F(jt)e^{-j\omega t}dt \;= \mathcal{F}[F(jt)]$$

Fourier Transform for Real Functions

If $f(t)$ is a real function, and $F(j\omega) = F_R(j\omega) + jF_I(j\omega)$

➡ $F(-j\omega) = F^*(j\omega)$

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

$$F^*(j\omega) = \int_{-\infty}^{\infty} f(t)e^{j\omega t}dt = F(-j\omega)$$

Fourier Transform for Real Functions

If $f(t)$ is a real function, and $F(j\omega) = F_R(j\omega) + jF_I(j\omega)$

➡ $F(-j\omega) = F^*(j\omega)$

➡ $F_R(j\omega)$ is even, and $F_I(j\omega)$ is odd.

$F_R(-j\omega) = F_R(j\omega)$ $F_I(-j\omega) = -F_I(j\omega)$

➡ *Magnitude spectrum* $|F(j\omega)|$ is even, and *phase spectrum* $\phi(\omega)$ is odd.

Fourier Transform for Real Functions

If $f(t)$ is real and even

➡ $F(j\omega)$ is real ✓

Pf)

Even ➡ $f(t) = f(-t)$

➡ $F(j\omega) = F(-j\omega)$

Real ➡ $F(-j\omega) = F^*(j\omega)$

➡ $F(j\omega) = F^*(j\omega)$

If $f(t)$ is real and odd

➡ $F(j\omega)$ is pure imaginary ✓

Pf)

Odd ➡ $f(t) = -f(-t)$

➡ $F(j\omega) = -F(-j\omega)$

Real ➡ $F(-j\omega) = F^*(j\omega)$

➡ $F(j\omega) = -F^*(j\omega)$

Example:

$$\mathcal{F}[f(t)] = F(j\omega) \qquad \mathcal{F}[f(t)\cos\omega_0 t] = ?$$

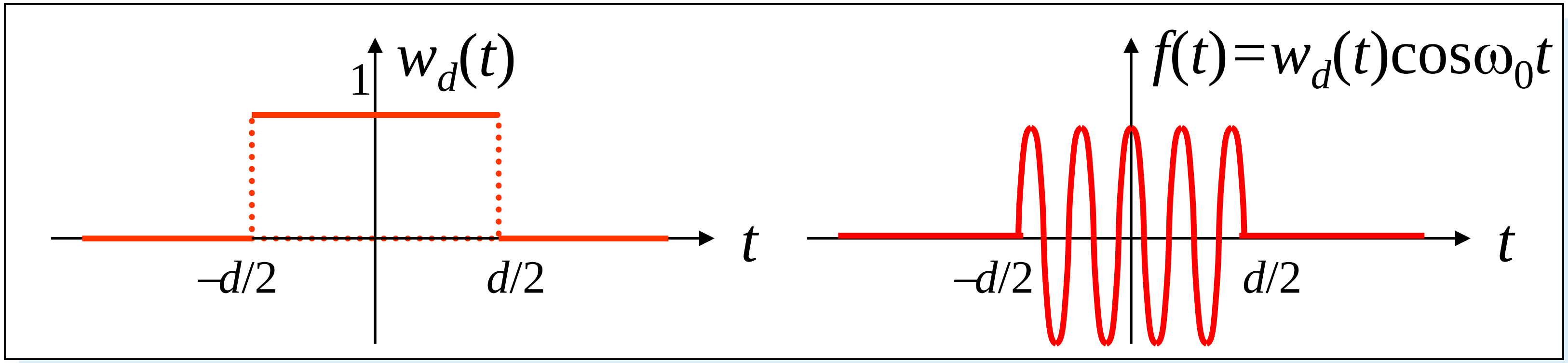
Sol)

$$f(t)\cos\omega_0 t = \frac{1}{2} f(t)(e^{j\omega_0 t} + e^{-j\omega_0 t})$$

$$\mathcal{F}[f(t)\cos\omega_0 t] = \frac{1}{2}\mathcal{F}[f(t)e^{j\omega_0 t}] + \frac{1}{2}\mathcal{F}[f(t)e^{-j\omega_0 t}]$$

$$= \frac{1}{2}F[j(\omega-\omega_0)] + \frac{1}{2}F[j(\omega+\omega_0)]$$

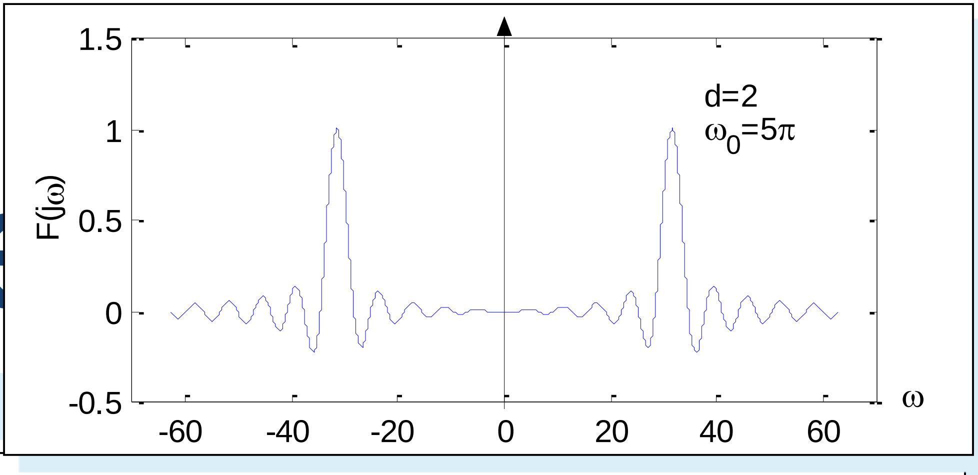
Example:

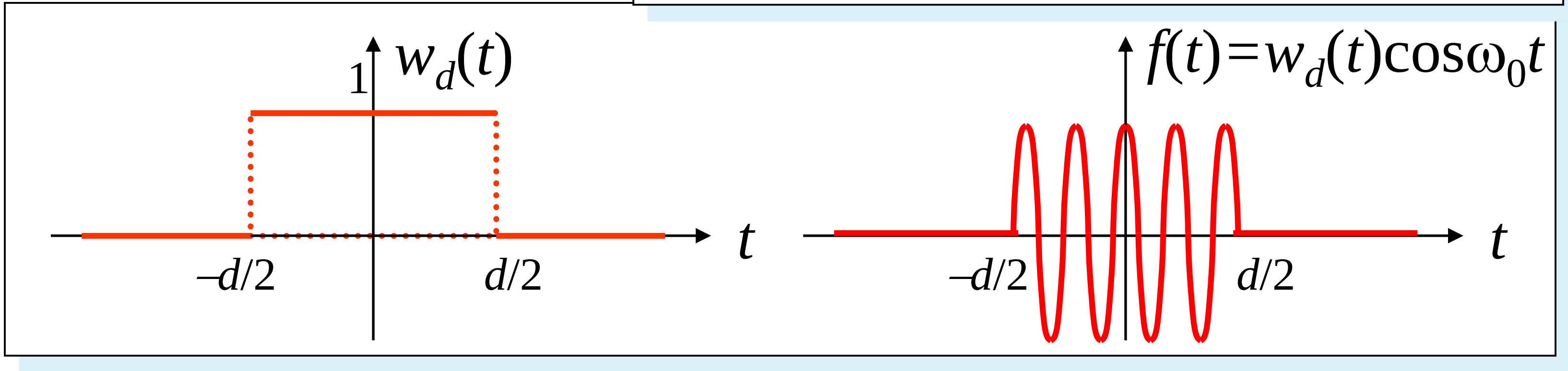


$$W_d(j\omega)=\mathcal{F}[w_d(t)]=\int_{-d/2}^{d/2}e^{-j\omega t}dt=\frac{2}{\omega}\sin\left(\frac{\omega d}{2}\right)=\frac{1}{\pi f}\sin\pi fd=d\sin c\pi fd$$

$$F(j\omega)=\mathcal{F}[w_d(t)\cos\omega_0 t]=\frac{\sin\frac{d}{2}(\omega-\omega_0)}{\omega-\omega_0}+\frac{\sin\frac{d}{2}(\omega+\omega_0)}{\omega+\omega_0}$$

Example

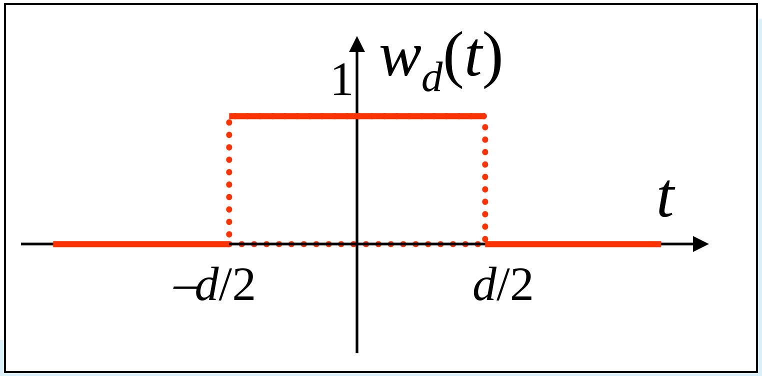




$$W_d(j\omega) = \mathcal{F}[w_d(t)] = \int_{-d/2}^{d/2} e^{-j\omega t} dt = \frac{2}{\omega} \sin\left(\frac{\omega d}{2}\right) = d \sin c \pi f d$$

$$F(j\omega) = \mathcal{F}[w_d(t) \cos \omega_0 t] = \frac{\sin \frac{d}{2}(\omega - \omega_0)}{\omega - \omega_0} + \frac{\sin \frac{d}{2}(\omega + \omega_0)}{\omega + \omega_0}$$

Example:



$$f(t) = \frac{\sin at}{\pi t} \qquad F(j\omega) = ?$$

Sol)

$$W_d(j\omega) = \frac{2}{\omega}\sin\left(\frac{\omega d}{2}\right)$$

$$\Rightarrow \mathcal{F}[W_d(jt)] = \mathcal{F}\left[\frac{2}{t}\sin\left(\frac{td}{2}\right)\right] = 2\pi w_d(-\omega)$$

$$\Rightarrow \mathcal{F}[f(t)] = \mathcal{F}\left[\frac{\sin at}{\pi t}\right] = w_{2a}(-\omega) = \begin{cases} 0 & \omega <|a| \\ 1 & \omega >|a| \end{cases}$$

About writing

$$\mathcal{F}[W_d(jt)] = \mathcal{F}\left[\frac{2}{t}\sin\left(\frac{td}{2}\right)\right] = 2\pi w_d(-\omega)$$

$$\mathcal{F}[f(t)] = \mathcal{F}\left[\frac{\sin at}{\pi t}\right] = w_{2a}(-\omega)$$

$$\mathcal{F}[f(t)] = \mathcal{F}\left[\frac{a}{\pi}\sin cat\right] = w_{2a}(-\omega) = \mathrm{Re}\,ct\frac{(\omega)}{[2a]} \equiv \mathrm{Re}\,c\frac{(f)}{[a/\pi]}$$

$$be\ carreful\ \mathrm{Re}\,ct\frac{(x)}{[\tau]} = \prod_{\left[-\frac{\tau}{2},\frac{\tau}{2}\right]}(x) = \begin{cases} 1 & if\ |x| \langle \tau/2 \\ 0 & otherwise \end{cases}$$

Fourier Transform of $f'(t)$

$$f(t) \xleftrightarrow{\mathcal{F}} F(j\omega) \quad \text{and if} \quad \lim_{t \to \pm\infty} f(t) = 0$$

$$\Rightarrow \quad f'(t) \xleftrightarrow{\mathcal{F}} j\omega F(j\omega)$$

Pf)

$$\mathcal{F}[f'(t)] = \int_{-\infty}^{\infty} f'(t) e^{-j\omega t} dt$$

$$= f(t) e^{-j\omega t} \Big|_{-\infty}^{\infty} + j\omega \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

$$= j\omega F(j\omega)$$

Fourier Transform of $f^{(n)}(t)$

$$f(t) \xleftrightarrow{\mathcal{F}} F(j\omega) \quad \text{and if} \quad \lim_{t\to\pm\infty} f(t) = 0$$

$$\Rightarrow \quad f^{(n)}(t) \xleftrightarrow{\mathcal{F}} (j\omega)^n F(j\omega)$$

Proved by yourselves

Fourier Transform of Integral

$$f(t) \overset{\mathcal{F}}{\longleftrightarrow} F(j\omega) \quad \text{and if} \quad \int_{-\infty}^{\infty} f(t)dt = F(0) = 0$$

$$\Rightarrow \quad \mathcal{F}\left[\int_{-\infty}^{t} f(x)dx\right] = \frac{1}{j\omega}F(j\omega)$$

Let $\phi(t) = \int_{-\infty}^{t} f(x)dx$ $\Rightarrow$ $\lim_{t\to\infty}\phi(t) = 0$

$$\mathcal{F}[\phi'(t)] = \mathcal{F}[f(t)] = F(j\omega) = j\omega\Phi(j\omega)$$

$$\Phi(j\omega) = \frac{1}{j\omega}F(j\omega)$$

(Suite)

General case

By convolution with heaviside distribution

$$\int_{-\infty}^{t} f(\tau)d\tau = \pi F(0)\delta(\omega) + \frac{1}{j\omega}F(j\omega)$$

The Derivative of Fourier Transform

$$\mathcal{F}[-jtf(t)] \overset{\mathcal{F}}{\longleftrightarrow} \frac{dF(j\omega)}{d\omega}$$

Pf)

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

$$\frac{dF(j\omega)}{d\omega} = \frac{d}{d\omega}\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt \;\; = \int_{-\infty}^{\infty} f(t)\frac{\partial}{\partial\omega}e^{-j\omega t}dt$$

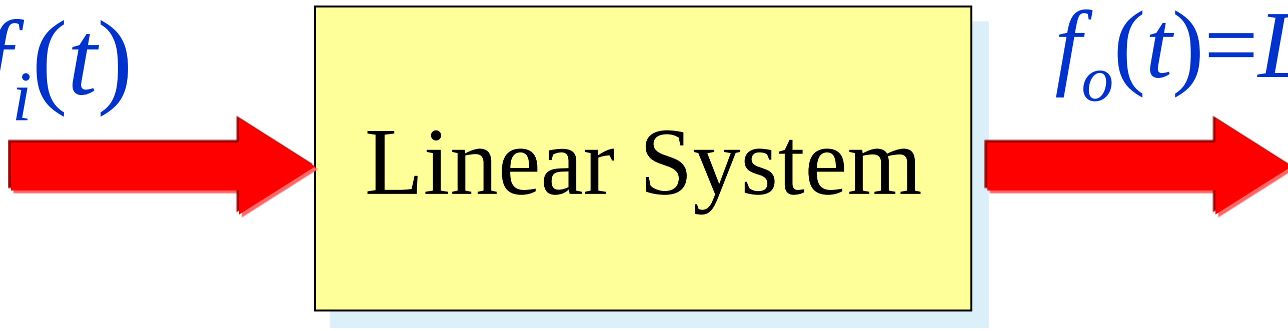
$$= \int_{-\infty}^{\infty} [-jtf(t)]e^{-j\omega t}dt \;\; = \mathcal{F}[-jtf(t)]$$

Continuous-Time Fourier Transform

Convolution

Basic Concept

$f_i(t)$



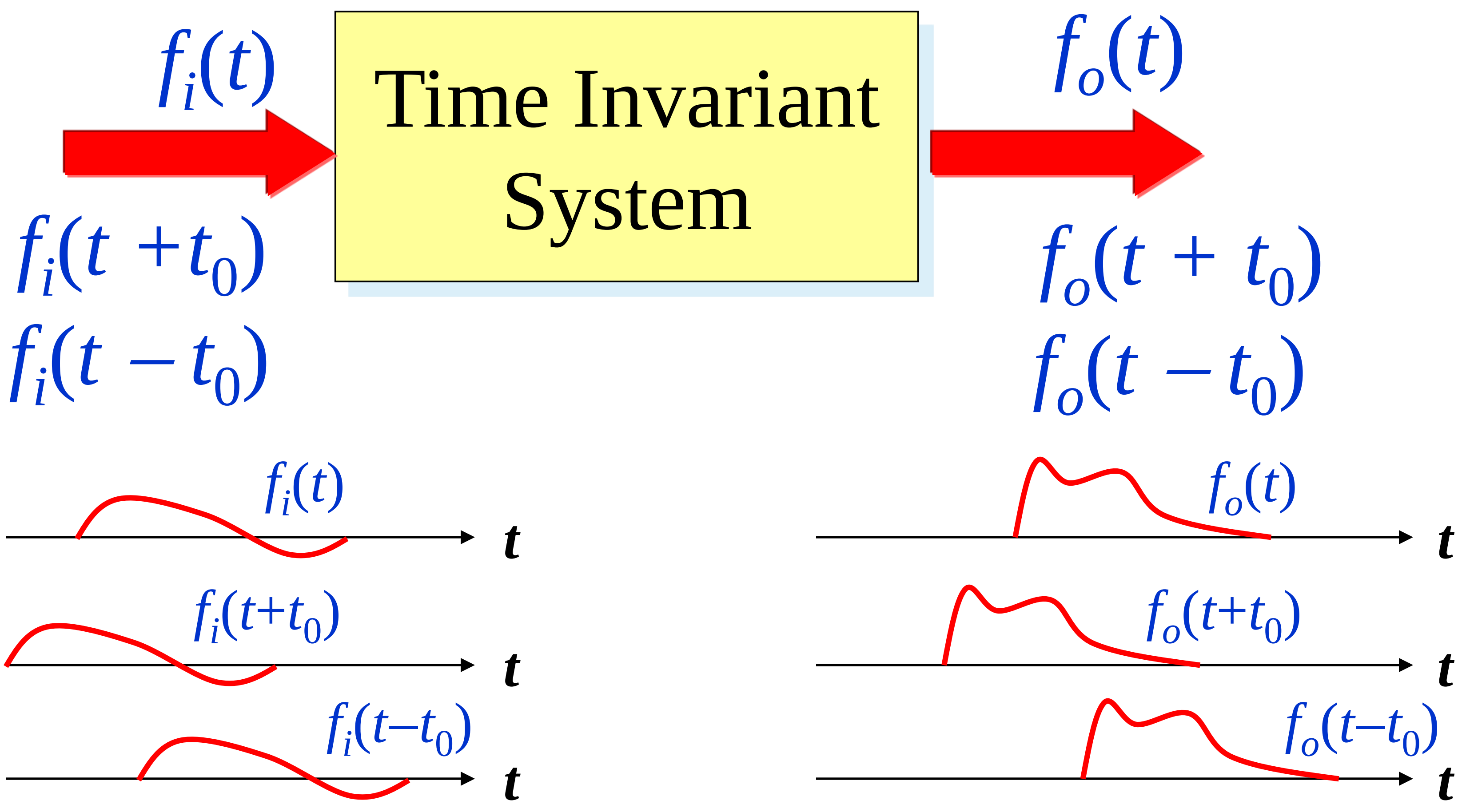
$f_o(t)=L[f_i(t)]$

$$f_i(t)=a_1\, f_{i1}(t) + a_2\, f_{i2}(t) \Rightarrow f_o(t)=L[a_1\, f_{i1}(t) + a_2\, f_{i2}(t)]$$

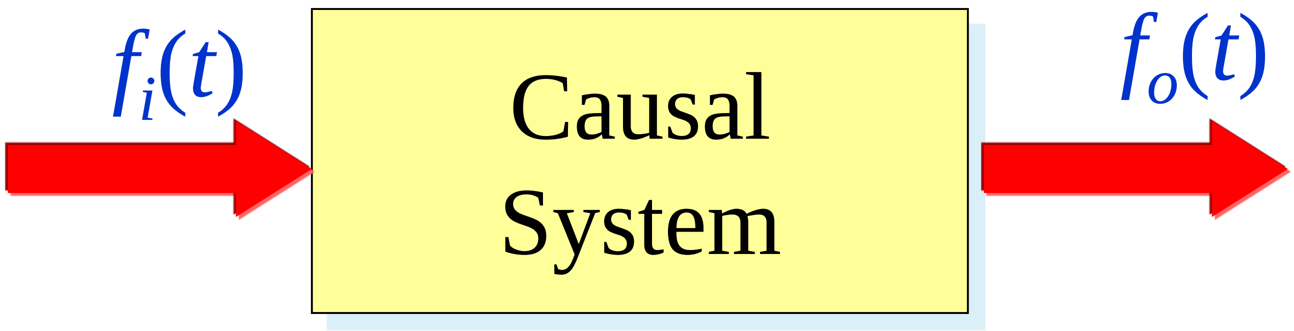
A *linear system* satisfies

$$f_o(t) = a_1 L[f_{i1}(t)] + a_2 L[f_{i2}(t)]$$
$$= a_1 f_{o1}(t) + a_2 f_{o2}(t)$$

Basic Concept



Basic Concept

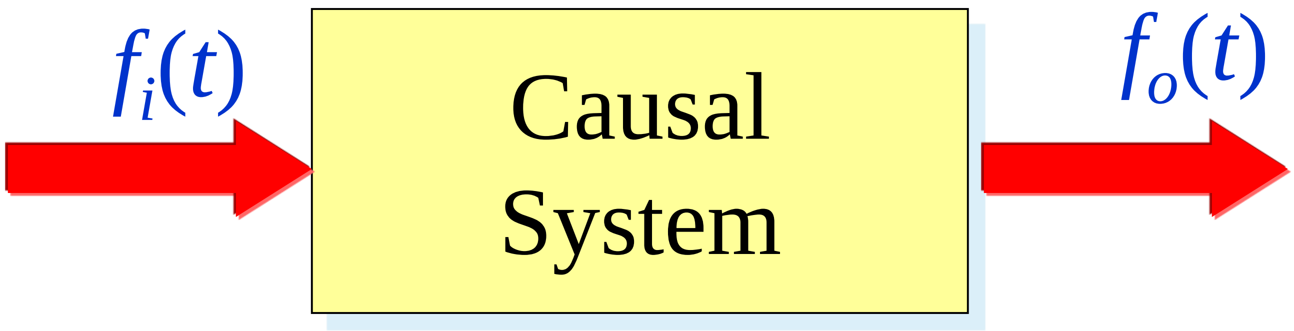


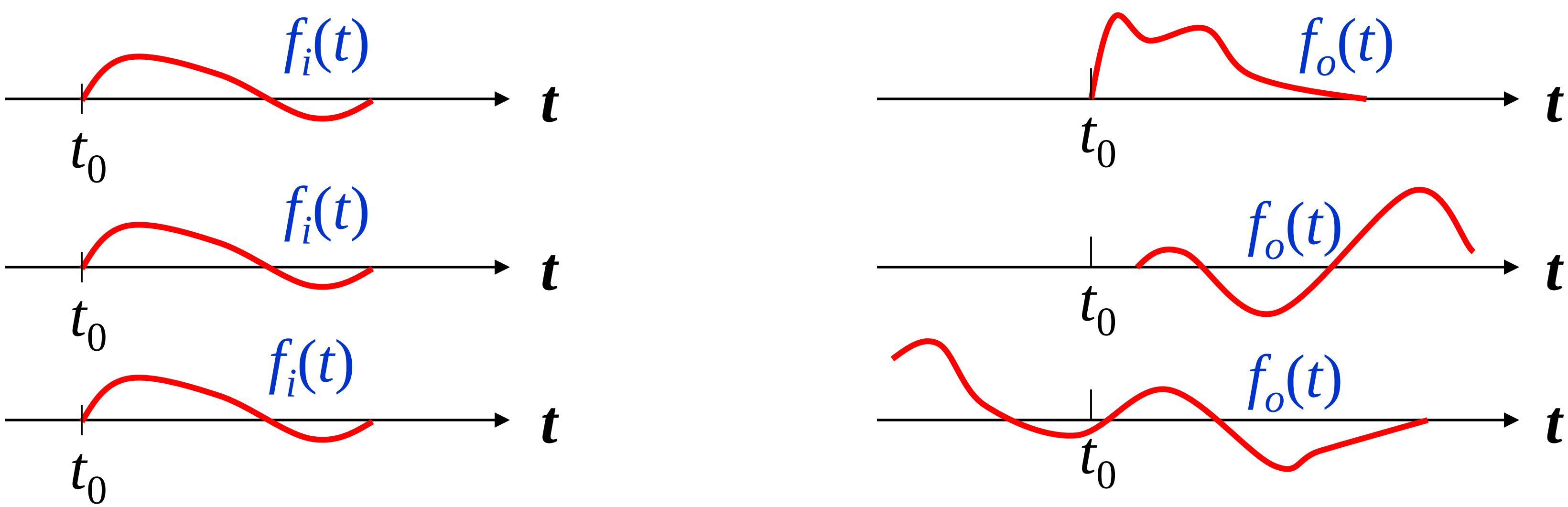
A *causal system* satisfies

$$f_i(t) = 0 \text{ for } t < t_0 \Rightarrow f_o(t) = 0 \text{ for } t < t_0$$

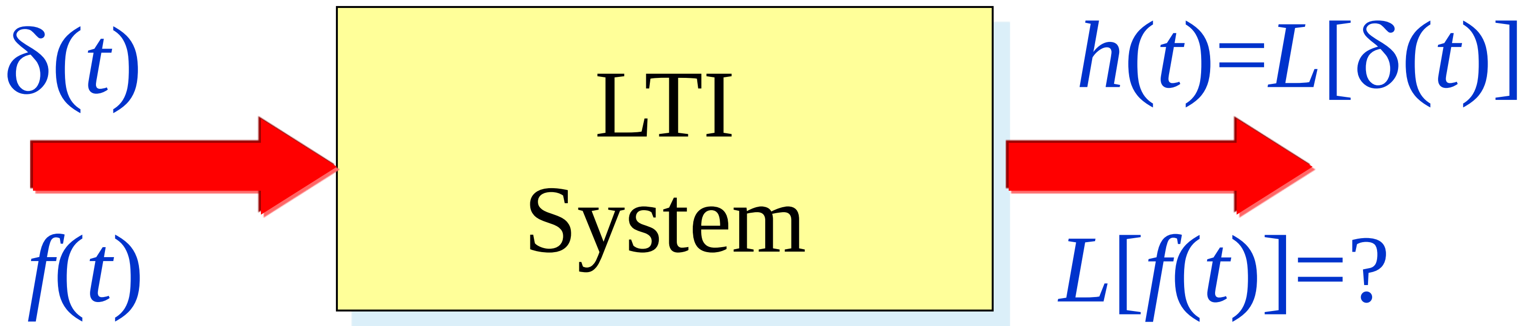
Which of the following systems are causal?

Basic Concept





Unit Impulse Response

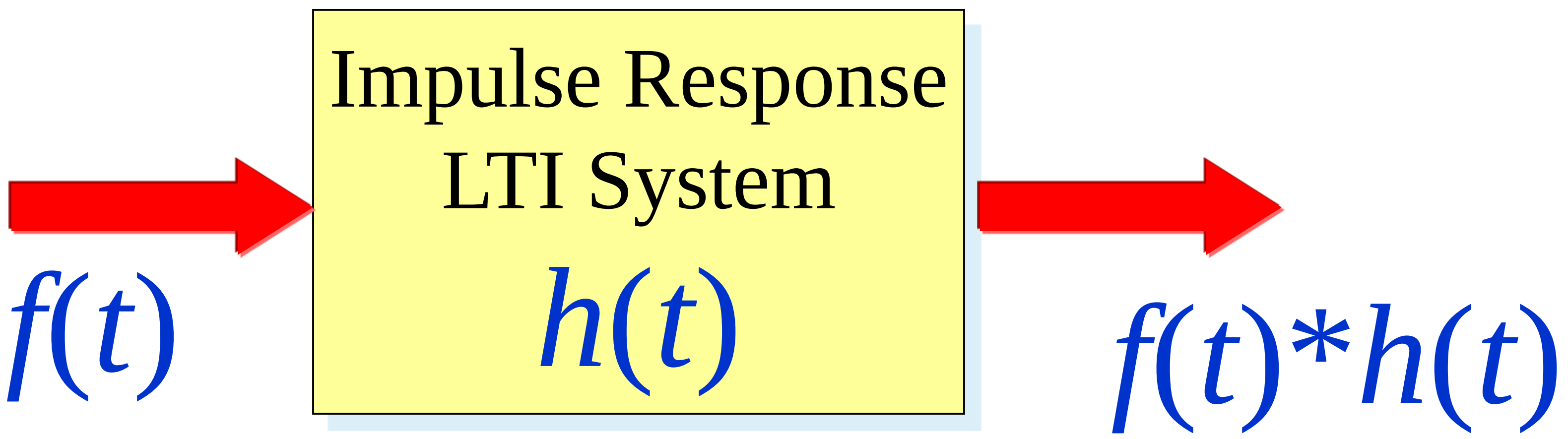


Facts: $\int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)d\tau = \int_{-\infty}^{\infty} f(t-\tau)\delta(\tau)d\tau = f(t)$

$$L[f(t)] = L\left[\int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)d\tau\right] = \int_{-\infty}^{\infty} f(\tau)L[\delta(t-\tau)]d\tau$$

$$= \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau$$ **Convolution**

Unit Impulse Response



$$L[f(t)] = f(t) * h(t)$$

Convolution Definition

The convolution of two functions $f_1(t)$ and $f_2(t)$ is defined as:

$$f(t) = \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau) d\tau$$

$$= f_1(t) * f_2(t)$$

Properties of Convolution

$$f_1(t) * f_2(t) = f_2(t) * f_1(t)$$

$$f_1(t) * f_2(t) = \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau) d\tau = \int_{\tau=-\infty}^{\tau=\infty} f_1(\tau) f_2(t-\tau) d\tau$$

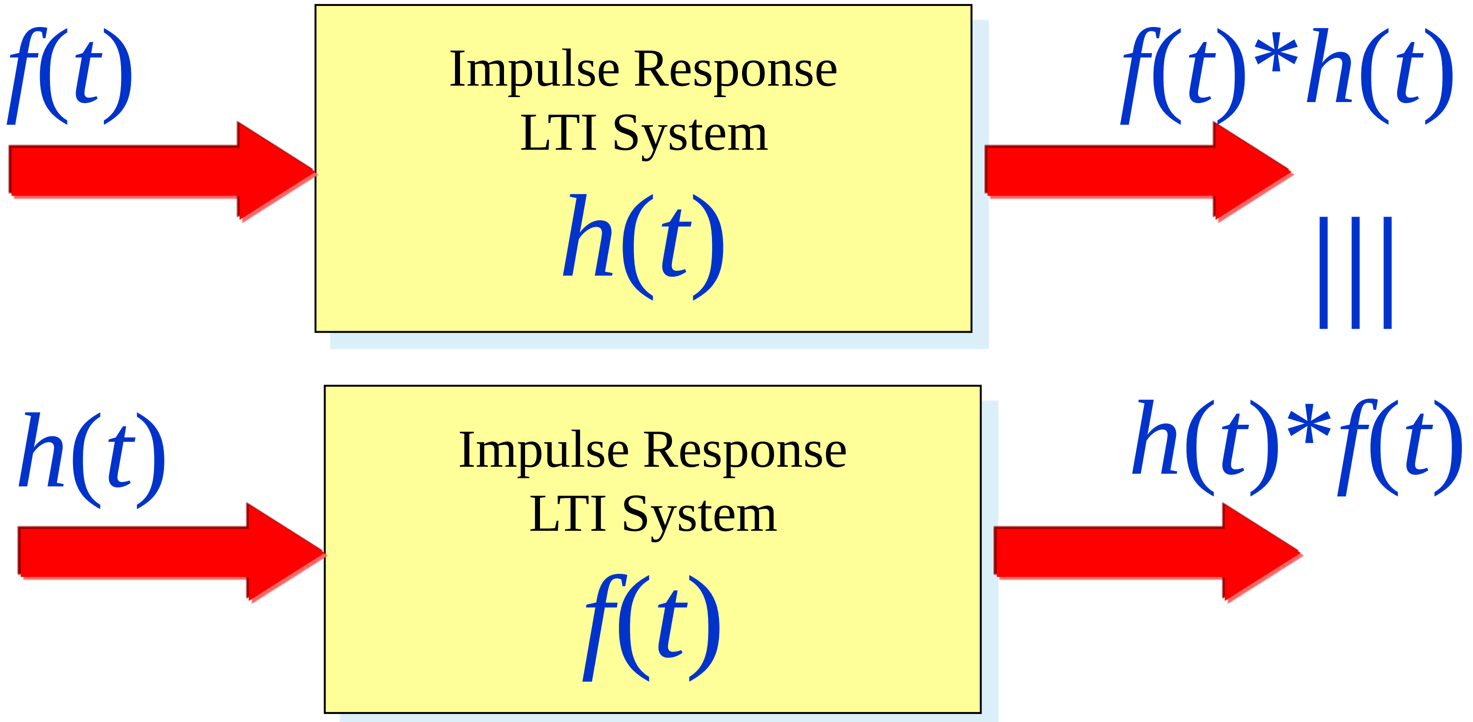
$$= \int_{t-\tau=-\infty}^{t-\tau=\infty} f_1(t-\tau) f_2[t-(t-\tau)] d(t-\tau)$$

$$= -\int_{\tau=\infty}^{\tau=-\infty} f_1(t-\tau) f_2(\tau) d\tau$$

$$= \int_{-\infty}^{\infty} f_1(t-\tau) f_2(\tau) d\tau \quad = f_2(t) * f_1(t)$$

Properties of Convolution

$$f_1(t) * f_2(t) = f_2(t) * f_1(t)$$



Properties of Convolution

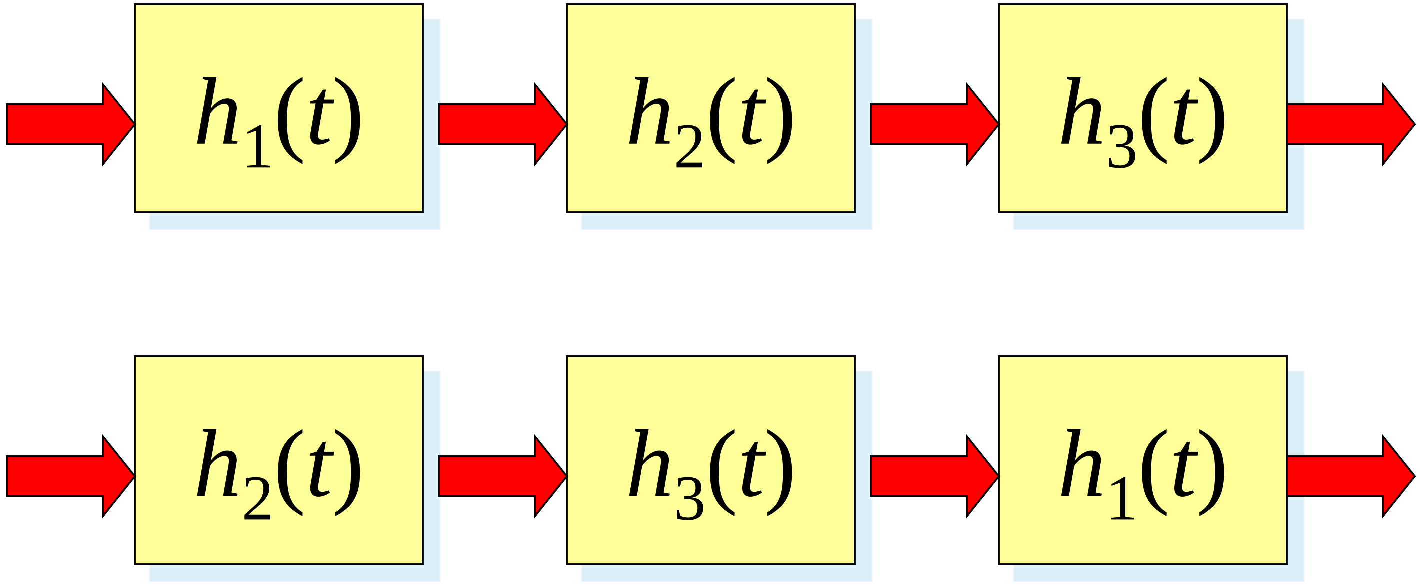
$$[f_1(t) * f_2(t)] * f_3(t) = f_1(t) * [f_2(t) * f_3(t)]$$

Prove by yourselves

The following two systems are identical

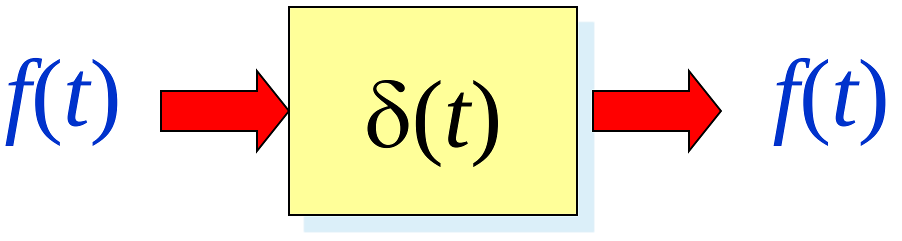
Properties of Convolution

$$[f_1(t) * f_2(t)] * f_3(t) = f_1(t) * [f_2(t) * f_3(t)]$$



Properties of Convolution

$$f(t) * \delta(t) = f(t)$$



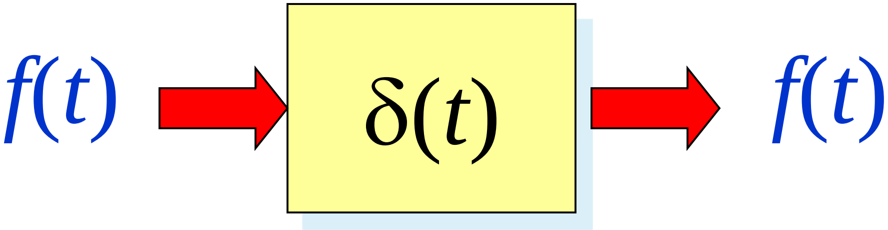
$$f(t) * \delta(t) = \int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)d\tau$$

$$= \int_{-\infty}^{\infty} f(t-\tau)\delta(\tau)d\tau$$

$$= f(t)$$

Properties of Convolution

$$f(t) * \delta(t) = f(t)$$



$$f(t) * \delta(t-T) = f(t-T)$$

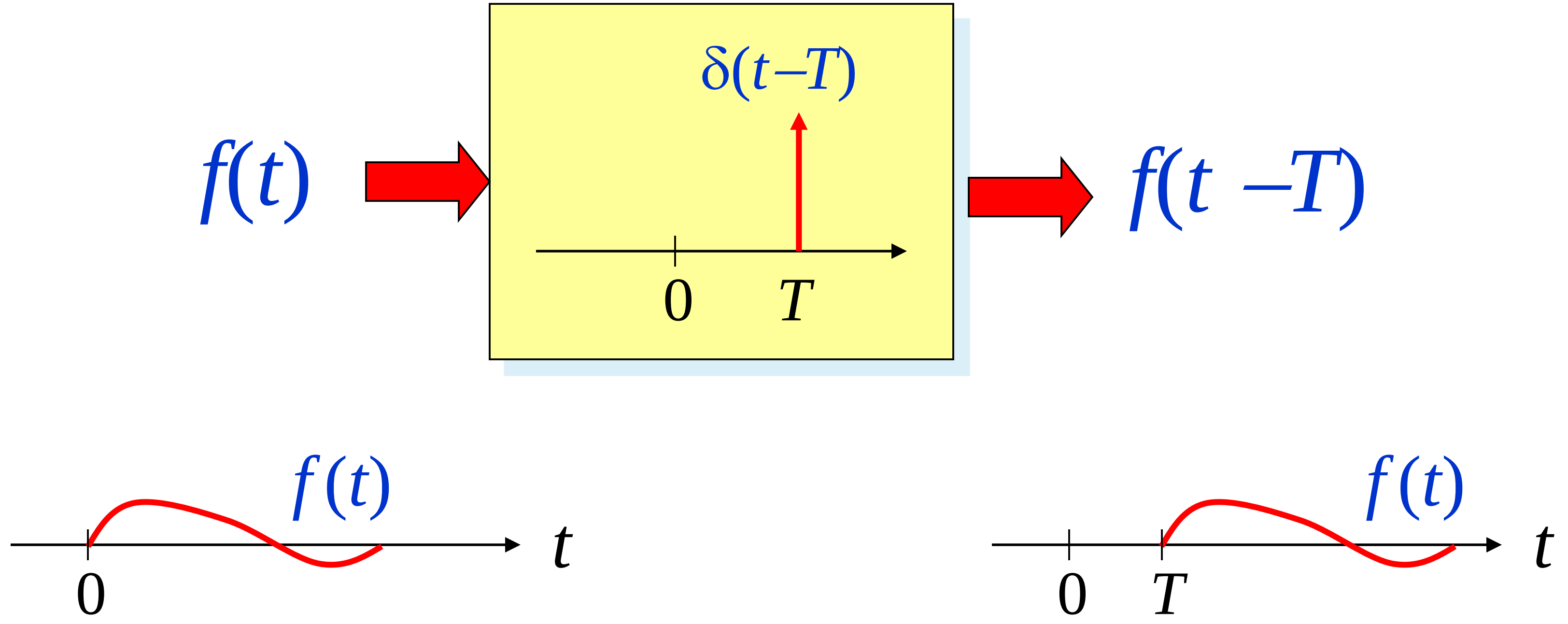
$$f(t) * \delta(t-T) = \int_{-\infty}^{\infty} f(\tau)\delta(t-T-\tau)d\tau$$

$$= \int_{-\infty}^{\infty} f(t-T-\tau)\delta(\tau)d\tau$$

$$= f(t-T)$$

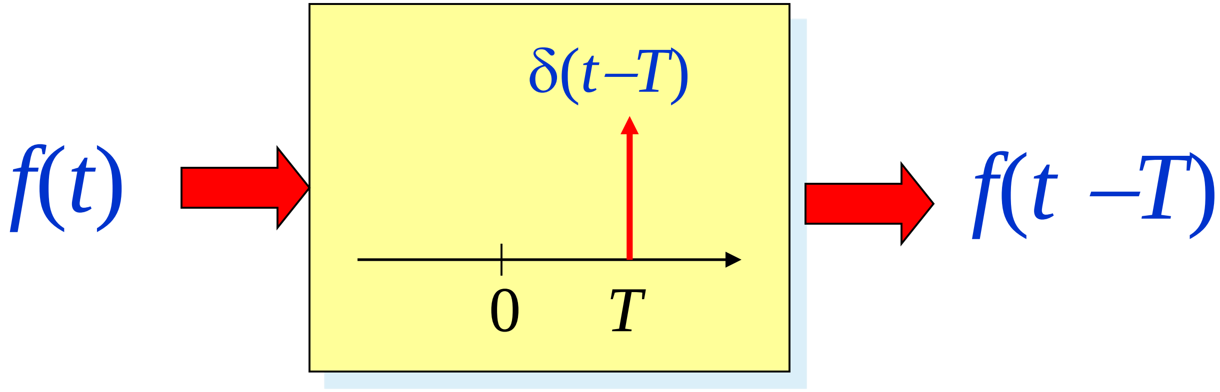
Properties of Convolution

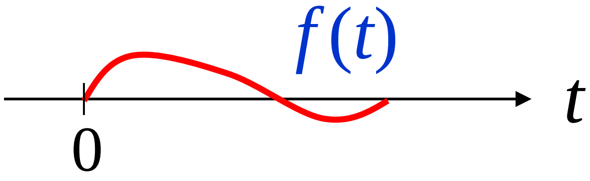
$$f(t) * \delta(t-T) = f(t-T)$$

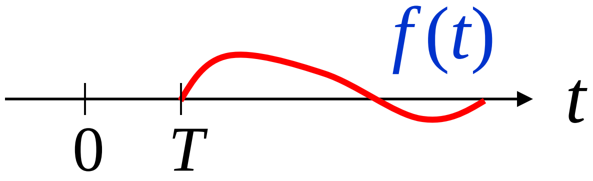


System function δ(*t*–*T*) serves as an *ideal delay* or a *copier*.

$$f(t) * \delta(t-T) = f(t-T)$$







Properties of Convolution

$$f_1(t) * f_2(t) \overset{\mathcal{F}}{\longleftrightarrow} F_1(j\omega)F_2(j\omega)$$

$$F[f_1(t) * f_2(t)] = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau)d\tau\right]e^{-j\omega t}dt$$

$$= \int_{-\infty}^{\infty} f_1(\tau)\left[\int_{-\infty}^{\infty} f_2(t-\tau)e^{-j\omega t}dt\right]d\tau$$

$$= \int_{-\infty}^{\infty} f_1(\tau)F_2(j\omega)e^{-j\omega\tau}d\tau$$

$$= F_2(j\omega)\int_{-\infty}^{\infty} f_1(\tau)e^{-j\omega\tau}d\tau \quad = F_1(j\omega)F_2(j\omega)$$

Time Domain	Frequency Domain
convolution	multiplication

$$f_1(t) * f_2(t) \overset{\mathcal{F}}{\longleftrightarrow} F_1(j\omega)F_2(j\omega)$$

$$F[f_1(t) * f_2(t)] = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau)d\tau\right]e^{-j\omega t}dt$$

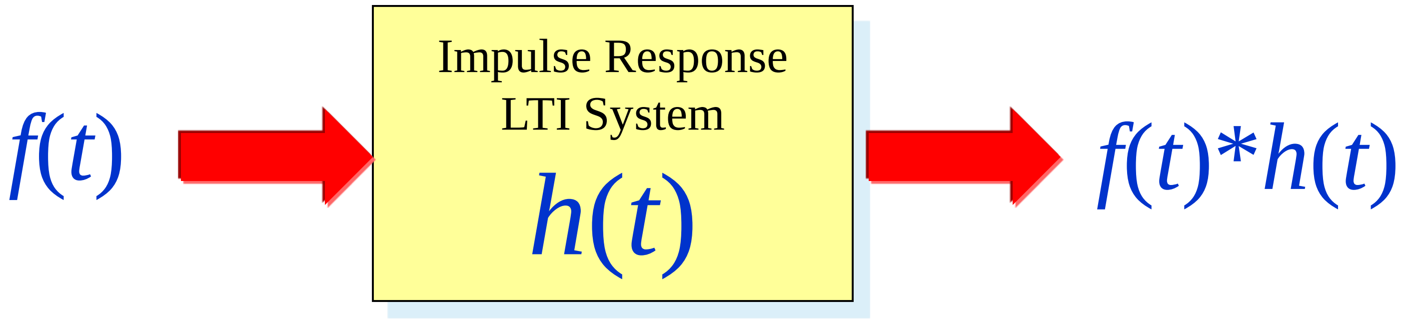
$$= \int_{-\infty}^{\infty} f_1(\tau)\left[\int_{-\infty}^{\infty} f_2(t-\tau)e^{-j\omega t}dt\right]d\tau$$

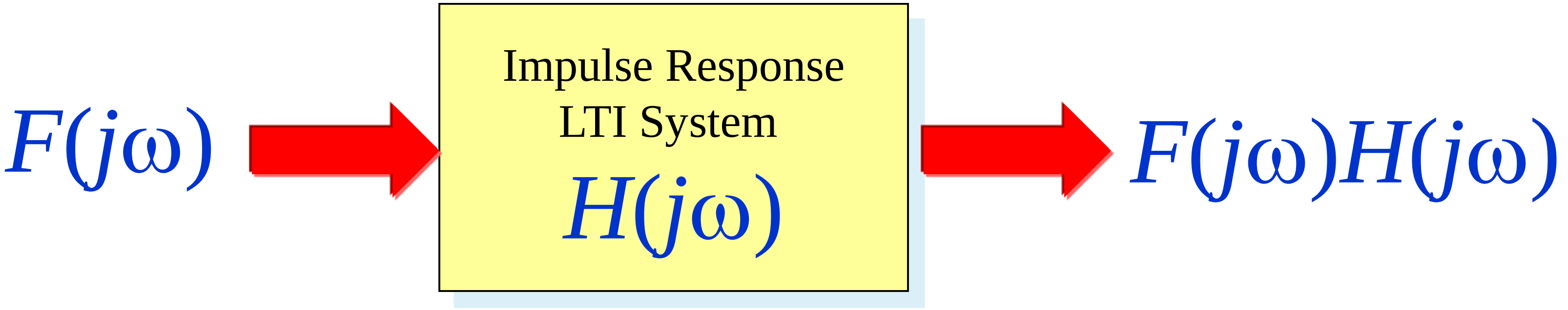
$$= \int_{-\infty}^{\infty} f_1(\tau)F_2(j\omega)e^{-j\omega\tau}d\tau$$

$$= F_2(j\omega)\int_{-\infty}^{\infty} f_1(\tau)e^{-j\omega\tau}d\tau \quad = F_1(j\omega)F_2(j\omega)$$

Time Domain	Frequency Domain
convolution	multiplication

$$f_1(t) * f_2(t) \overset{\mathcal{F}}{\longleftrightarrow} F_1(j\omega)F_2(j\omega)$$

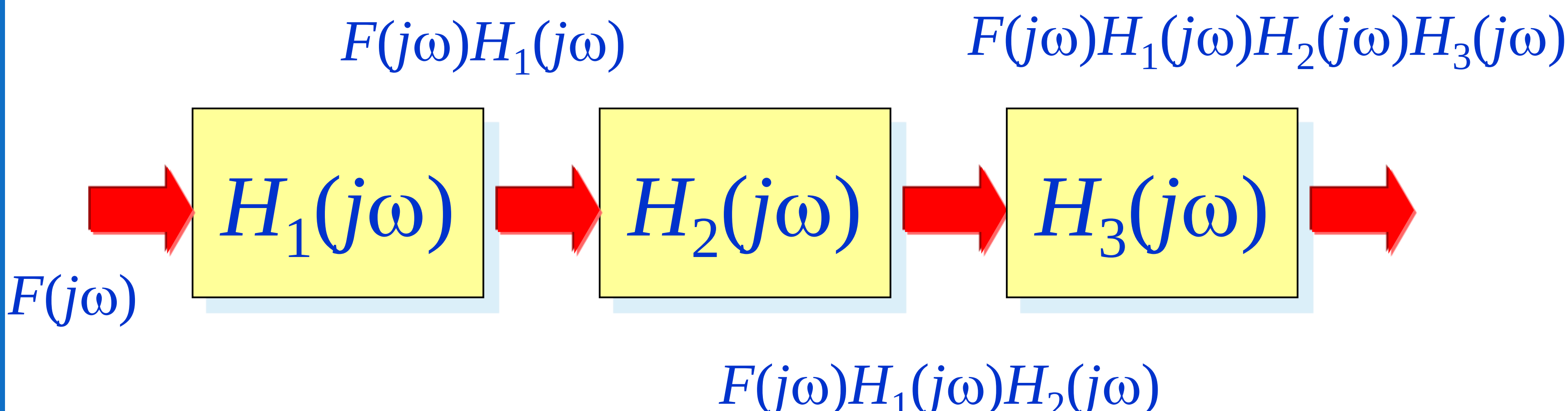




Time Domain | **Frequency Domain**

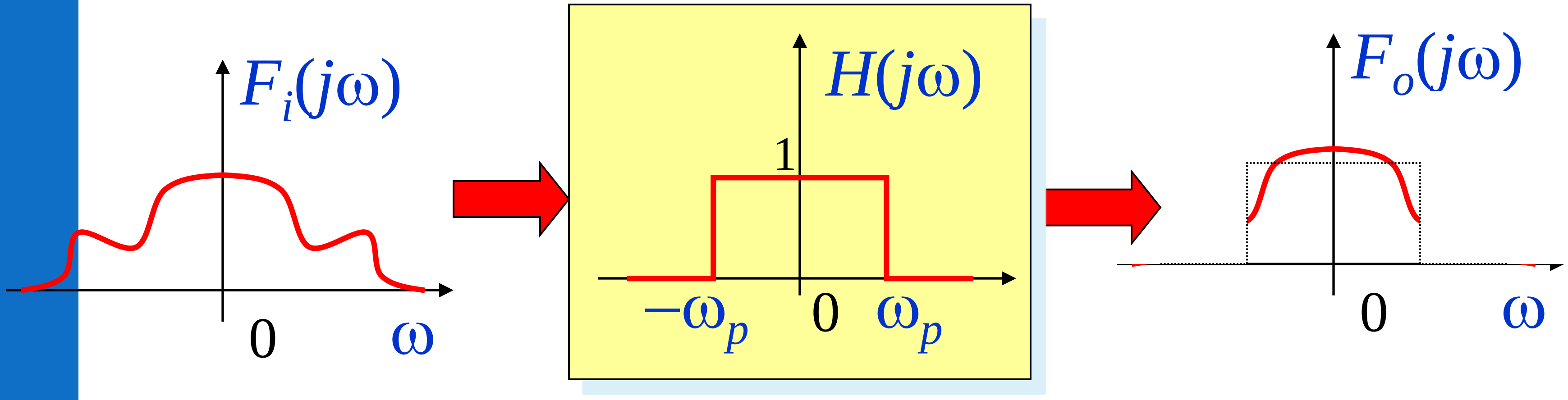
convolution | **multiplication**

$$f_1(t) * f_2(t) \overset{\mathcal{F}}{\longleftrightarrow} F_1(j\omega)F_2(j\omega)$$



Properties of Convolution

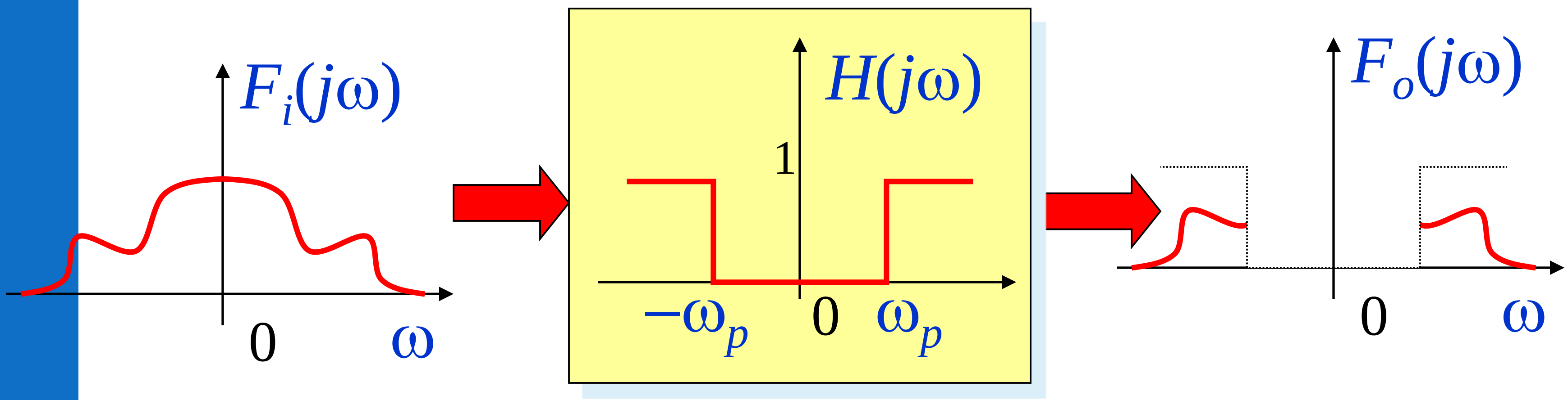
$$f_1(t) * f_2(t) \overset{\mathcal{F}}{\longleftrightarrow} F_1(j\omega)F_2(j\omega)$$



An Ideal Low-Pass Filter

Properties of Convolution

$$f_1(t) * f_2(t) \overset{\mathcal{F}}{\longleftrightarrow} F_1(j\omega)F_2(j\omega)$$



An Ideal High-Pass Filter

Properties of Convolution

$$f_1(t)\,f_2(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(j\theta)F_2[j(\omega-\theta)]d\theta$$

$$f_1(t)\,f_2(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2\pi} F_1(j\omega) * F_2(j\omega)$$

Prove by yourselves

Time Domain	Frequency Domain
multiplication	convolution

$$f_1(t)\,f_2(t) \xleftrightarrow{\mathcal{F}} \frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(j\theta)F_2[j(\omega-\theta)]d\theta$$

$$f_1(t)\,f_2(t) \xleftrightarrow{\mathcal{F}} \frac{1}{2\pi}F_1(j\omega) * F_2(j\omega)$$

Prove by yourselves

Continuous-Time Fourier Transform

Parseval's Theorem

Properties of Convolution

$$\int_{-\infty}^{\infty}[f_1(t)f_2(t)]dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}F_1(j\omega)F_2[-j\omega]d\omega$$

$$f_1(t)f_2(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2\pi}\int_{-\infty}^{\infty}F_1(j\theta)F_2[j(\omega-\theta)]d\theta$$

$$\int_{-\infty}^{\infty}[f_1(t)f_2(t)]e^{j\omega t}dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}F_1(j\theta)F_2[j(\omega-\theta)]d\theta$$

$\omega=0$

$$\int_{-\infty}^{\infty}[f_1(t)f_2(t)]dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}F_1(j\omega)F_2[j(-\omega)]d\omega$$

Properties of Convolution

$$\int_{-\infty}^{\infty}[f_1(t)f_2(t)]dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}F_1(j\omega)F_2[-j\omega]d\omega$$

If $f_1(t)$ and $f_2(t)$ are real functions,

$$\int_{-\infty}^{\infty}[f_1(t)f_2(t)]dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}F_1(j\omega)F_2^*[j\omega]d\omega$$

$f_2(t)$ real $\Rightarrow$ $F_2[-j\omega] = F_2^*[j\omega]$

Parseval's Theorem: Energy Preserving

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(j\omega)|^2 d\omega$$

$$\mathcal{F}[f^*(t)] = \int_{-\infty}^{\infty} f^*(t)e^{-j\omega t}dt = \left(\int_{-\infty}^{\infty} f(t)e^{j\omega t}dt\right)^* = F^*(-j\omega)$$

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} f(t)f^*(t)dt = Energy\ of\ \ f(t)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)F^*[-(-j\omega)]d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(j\omega)|^2 d\omega$$

Isometry

$$\left\langle f_1(t), f_2(t)\right\rangle = \frac{1}{2\pi}\left\langle F_1(\omega), F_2(\omega)\right\rangle = \left\langle F_1(\upsilon), F_2(\upsilon)\right\rangle$$

$$\left\| f(t)\right\|_{L^2}^2 = \frac{1}{2\pi}\left\| F(\omega)\right\|_{L^2}^2 = \left\| F(\upsilon)\right\|_{L^2}^2$$