Course L3
Signal theory

Correlation and spectral density functions

The correlation or cross correlation function is one of the main tools for the measurement of likethehood between two functions which represent a physical phenomenom. Considering analogic and discrete signals, the objective of this course is to give the main results of this tool.
The latter is used in electronics, telecommunications… for the detection of targets, sources in aeronautics or astronomy…, everytime we have to detect a potential source.

analogic signals

1st definition :

If $s(t) \in L^2$

$$C_s(\tau) = \int_{-\infty}^{\infty} s(t)\bar{s}(t-\tau)dt$$

$$C_s(0) = \text{Energy of } s(t) = \|s(t)\|_2^2$$

2nd definition :

If $s(t) \notin L^2$

$$C_s(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} s(t)\bar{s}(t-\tau)dt$$

$$C_s(0) = \text{Power of } s(t)$$

$$= C_s(0) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |s(t)|^2 dt$$

3rd definition :

If $s(t)$ is periodic with period T_0

$$C_s(\tau) = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} s(t)\bar{s}(t-\tau)dt$$

$C_s(0) = \text{Power of } s(t) =$

$$C_s(0) = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} |s(t)|^2 dt$$

s(t) can be decomposed in a Fourier series

with $s(t) = \sum_{n=-\infty}^{\infty} C_n e^{j2\pi nt/T_0}$ and:

discrete signals

$$C_s(m) = \sum_{n\in\mathbb{Z}} s(n)\bar{s}(n-m)$$

$$C_s(0) = \text{Energy of s(n)=} \quad C_s(m) = \sum_{n\in\mathbb{Z}} |s(n)|^2$$

$$C_s(m) = \lim_{N\to\infty} \frac{1}{2N+1}\sum_{-N}^{N} s(n)\bar{s}(n-m)$$

$$C_s(0) = \text{Power of s(n)}$$

$$= C_s(0) = \lim_{N\to\infty} \frac{1}{2N+1}\sum_{-N}^{N} |s(n)|^2$$

$$C_s(m) = \frac{1}{N}\sum_{n=0}^{N-1} s(n)\bar{s}(n-m)$$

$C_s(0) = \text{Power of } s(n) =$

$$C_s(0) = \frac{1}{N}\sum_{n=0}^{N-1} |s(n)|^2$$

s(n) discrete and periodic, then,

$s(n) = \frac{1}{N}\sum_{k=0}^{N-1} S_k e^{j2\pi kn/N}$ and

$$C_n = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} s(t)e^{-j2\pi nt/T_0}dt$$

$$S_k = \sum_{k=0}^{N-1} s(n)e^{-2j\pi nk/N}$$

then :

$$C_s(\tau) = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} s(t)\bar{s}(t-\tau)dt \text{ can be written}$$

$$C_s(\tau) = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} \sum_{n\in\mathbb{Z}} C_n e^{j2\pi nt/T_0} \sum_{m\in\mathbb{Z}} \bar{C}_m e^{-j2\pi m(t-\tau)/T_0}$$

$$C_s(\tau) = \sum_{n,m\in\mathbb{Z}^2} C_n\bar{C}_m \underbrace{\frac{1}{T_0}\int_{t_0}^{t_0+T_0} e^{j2\pi t(n-m)/T_0}dt}_{\delta n,m} \quad e^{j2\pi m\tau/T_0}$$

$$C_s(m) = \frac{1}{N}\sum_{n=0}^{N-1} \frac{1}{N}\sum_{k=0}^{N-1} S_k e^{j2\pi kn/N} \frac{1}{N}\sum_{l=0}^{N-1} \bar{S}_l e^{-j2\pi l(n-m)/N}$$

$$C_s(m) = \frac{1}{N^3}\sum_{k,l=0}^{N-1,N-1} S_k\bar{S}_l \underbrace{\sum_{n=0}^{N-1} e^{j2\pi n(k-l)/N}}_{N\delta_{k,l}} e^{j2\pi lm/N}$$

$$C_s(m) = \frac{1}{N^2}\sum_{k=0}^{N-1} |S_k|^2 e^{j2\pi km/N} \text{ en posant } \alpha_k = \frac{S_k}{N}$$

It becomes :

$$C_s(\tau) = \sum_{n\in\mathbb{Z}} |C_n|^2 e^{j2\pi n\tau/T_0} \text{ et}$$

$$C_s(0) = \sum_{n\in\mathbb{Z}} |C_n|^2 = \text{Power}$$

$$C_s(m) = \sum_{k=0}^{N-1} |\alpha_k|^2 e^{j2\pi km/N}$$

$$C_s(0) = \sum_{k=0}^{N-1} |\alpha_k|^2 = \text{Power}$$

Example of computation :

$s(t) = H(t)e^{-at}$ $\quad a\rangle 0$ and H(t) : Heaviside

then : $C_s(\tau) = \dfrac{e^{-a|\tau|}}{2a}$

A few properties :
Using the first definition for written simplification

1-if s(t) is real :

$$C_s(\tau) = \int_{-\infty}^{\infty} s(t)s(t-\tau)dt$$

$$C_s(m) = \sum_{n\in\mathbb{Z}} s(n)s(n-m)$$

And $C_s(\tau) = C_s(-\tau)$ even function

And $C_s(m) = C_s(-m)$ even function

2-If s(t) is a derivable function :

$$\left.\frac{dC_s(\tau)}{d\tau}\right|_{\tau=0} = 0$$

3- $C_s(0)$ is maximum : $C_s(0) \geq C_s(\tau)$

Idem…

Writing :

$$\int_{-\infty}^{\infty} \left(s(t) \pm s(t-\tau)\right)^2 dt \geq 0$$

$$2C_s(0) \pm 2C_s(\tau) \geq 0$$

We get :

$$C_s(0) \geq |C_s(\tau)|$$

4- If $C_s(\tau)$ is continuous at origin then

$C_s(\tau)$ continuous everywhere : i.e.

$$\lim_{\tau\to 0} C_s(\tau) - C_s(0) = 0 \Leftrightarrow \forall \tau, \quad \lim_{\varepsilon\to 0} C_s(\tau+\varepsilon) - C_s(\tau) = 0$$

$$\lim_{\varepsilon\to 0} C_s(\tau+\varepsilon) - C_s(\tau) =$$

$$\lim_{\varepsilon\to 0} \int s(t)\left[s(t-\tau-\varepsilon) - s(t-\tau)\right]dt$$

The integral can be majored by :

$$\left[\int s(t)\left[s(t-\tau-\varepsilon) - s(t-\tau)\right]dt\right] \le$$

$$\left[\int \left(s(t)\right)^2 dt \int \left[s(t-\tau-\varepsilon) - s(t-\tau)\right]^2 dt\right]^{1/2}$$

Thus :

$$\left[2C_s(0)(C_s(0) - C(\varepsilon))\right]^{1/2}$$

Where the limit tends to 0 with ε

5- Relationship between convolution and correlation

If $s(t) \in L^2$, then :

$$C_s(\tau) = s(t) * s(-t)\big|_{t=\tau} = s(\tau) * s(-\tau)$$

$$C_s(m) = s(n) * s(-n)\big|_{n=m} = s(m) * s(-m)$$

Relation in the frequential domaine

Using the 1st definition and Parseval's theorem:

$$C_s(\tau) = \int_{-\infty}^{\infty} s(t)\bar{s}(t-\tau)dt = \int_{-\infty}^{\infty} \hat{s}(\upsilon)\bar{\hat{s}}(\upsilon)e^{j2\pi\upsilon\tau}d\upsilon$$

Wiener-Kinchin's theorem

$$\mathcal{F}\left(C_s(\tau)\right) = S_s(\upsilon)$$

Wiener-Kinchin's theorem

$$\mathcal{F}\left(C_s(m)\right) = S_s(\upsilon)$$

With the first definition

$$\mathcal{F}\left(C_s(\tau)\right) = \int_{-\infty}^{\infty} C_s(\tau)e^{-j2\pi\upsilon\tau}d\tau =$$

$$\int_{\mathbb{R}^2} s(t)\bar{s}(t-\tau)e^{-j2\pi\upsilon\tau}dtd\tau =$$

$$\int_{-\infty}^{\infty} s(t)\int_{-\infty}^{\infty} \bar{s}(t-\tau)e^{j2\pi\upsilon(t-\tau)}d\tau e^{-j2\pi\upsilon t}dt = \left|\hat{s}(\upsilon)\right|^2$$

$$\mathcal{F}\left(C_s(m)\right) = \sum_{-\infty}^{\infty} C_s(m)e^{-j2\pi m\upsilon} =$$

$$\sum_{n,m\in\mathbb{Z}^2} s(n)\bar{s}(n-m)e^{-j2\pi m\upsilon} =$$

$$\sum_{\mathbb{Z}} s(n)\sum_{\mathbb{Z}} \bar{s}(n-m)e^{j2\pi(n-m)\upsilon}e^{-j2\pi n\upsilon} = \left|\hat{s}(\upsilon)\right|^2$$

With the second definition, we get :

$$S_s(\upsilon) = \lim_{T\to\infty} \frac{1}{2T}\left|\int_{-T}^{T} s(t)e^{-j2\pi\upsilon t}dt\right|^2$$

$$\mathcal{F}\left(C_s(m)\right) = \mathcal{F}\left(\lim_{N\to\infty} \frac{1}{2N+1}\sum_{-N}^{N} s(n)\bar{s}(n-m)\right) \quad if \quad the \lim \exists$$

$$with \quad s(t) = \lim_{T\to\infty} s_T(t) = \lim_{T\to\infty} s(t)\operatorname{Re}ct\frac{(t)}{[T]}$$

$$and \quad C_s(\tau) = \lim_{T\to\infty} \frac{1}{2T} s_T(t) * s_T(-t)\,|_{t=\tau}$$

With the third definition

$$\mathcal{F}\left(C_s(\tau)\right) = \sum_{-\infty}^{\infty} \left|C_n\right|^2 \delta\left(\upsilon - \frac{n}{T_0}\right)$$

$$\mathcal{F}\left(C_s(m)\right) = \sum_{k=0}^{N-1} \left|\alpha_k\right|^2 \delta\left(\upsilon - \frac{k}{N}\right) \quad ; \alpha_k = \frac{S_k}{N}$$

A few words about cross correlation and cross spectral density function

For example, using the 1st definition, we can write :

$$C_{xy}(\tau) = \int_{-\infty}^{\infty} x(t)\overline{y}(t-\tau)dt$$

$$C_{xy}(m) = \sum_{n\in\mathbb{Z}} x(n)\overline{y}(n-m)$$

In the other case, when we can't write this integral,

$$C_{xy}(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)\overline{y}(t-\tau)dt$$

$$C_{xy}(m) = \lim_{N\to\infty} \frac{1}{2N+1}\sum_{-N}^{N} x(n)\overline{y}(n-m)$$

Also, when the 2 functions have the same periode T_0, we can put :

$$C_{xy}(\tau) = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)\overline{y}(t-\tau)dt$$

$$C_{xy}(m) = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\overline{y}(n-m)$$

About spectral density functions

1st definition

$$\mathscr{F}\left(C_{xy}(\tau)\right) = \int_{-\infty}^{\infty} C_{xy}(\tau)e^{-j2\pi\upsilon\tau}d\tau =$$

$$\int_{\mathbb{R}^2} x(t)\overline{y}(t-\tau)e^{-j2\pi\upsilon\tau}dtd\tau =$$

$$\int_{-\infty}^{\infty} x(t)\int_{-\infty}^{\infty} \overline{y}(t-\tau)e^{j2\pi\upsilon(t-\tau)}d\tau e^{-j2\pi\upsilon t}dt = \hat{x}(\upsilon)\overline{\hat{y}}(\upsilon)$$

$$\mathscr{F}\left(C_{xy}(m)\right) = \sum_{-\infty}^{\infty} C_{xy}(m)e^{-j2\pi m\upsilon} =$$

$$\sum_{n,m\in\mathbb{Z}^2} x(n)\overline{y}(n-m)e^{-j2\pi m\upsilon} =$$

$$\sum_{\mathbb{Z}} x(n)\sum_{\mathbb{Z}} \overline{y}(n-m)e^{j2\pi(n-m)\upsilon}e^{-j2\pi n\upsilon} = \hat{x}(\upsilon)\overline{\hat{y}}(\upsilon)$$

2nd definition

$$S_{xy}(\upsilon) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)e^{-j2\pi\upsilon t}dt\int_{-T}^{T} \overline{y}(t)e^{j2\pi\upsilon t}dt$$

$$with \quad x(t) = \lim_{T\to\infty} x_T(t) = x(t)\,\mathrm{Rect}\frac{(t)}{[T]}$$

$$and \quad C_{xy}(\tau) = \lim_{T\to\infty}\frac{1}{2T}x_T(t) * \overline{y}_T(-t)\,|_{t=\tau}$$

$$\mathscr{F}\left(C_{xy}(m)\right) = \mathscr{F}\left(\lim_{N\to\infty}\frac{1}{2N+1}\sum_{-N}^{N} x(n)\overline{y}(n-m)\right) \quad if \quad the \lim \exists$$

3rd definition

$$\mathscr{F}\left(C_{xy}(\tau)\right) = \sum_{-\infty}^{\infty} X_n\overline{Y}_n\delta\left(\upsilon - \frac{n}{T_0}\right)$$

$$\mathscr{F}\left(C_{xy}(m)\right) = \sum_{k=0}^{N-1} \alpha_k\overline{\beta}_k\delta\left(\upsilon - \frac{k}{N}\right)$$

With $\alpha_k = \frac{X_k}{N}$ and $\beta_k = \frac{Y_k}{N}$