Course L3
Signal theory

The Hilbert transform and analytic signals

Definition

If $s(t)$ is real, the Hilbert's transform of $s(t)$ is :

$$H_i\left[s(t)\right] = \sigma(t) = \left[VP\frac{1}{\pi u} * s(u)\right]_{u=t} = VP\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{s(u)}{t-u}du$$

Which converges for almost all t for $s \in L^p \quad (1 < p < \infty)$
This integral is considered in a Cauchian sense and the computation has to be done in the complex plane with the residual method.

Relation in the frequency domain

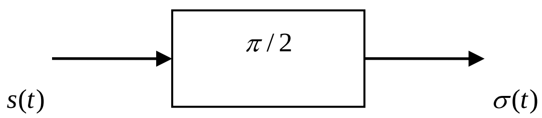
If we call : $\hat{s}(\upsilon)$ the Fourier transform of $s(t)$ and $\hat{\sigma}(\upsilon)$ the Fourier transform of $\sigma(t)$, then :

we know that : if $\begin{matrix} s(t) \leftrightarrow \hat{s}(\upsilon) \\ \hat{s}(t) \leftrightarrow s(-\upsilon) \end{matrix}$ then if $\mathscr{F}(\text{sgn}(t)) = \frac{1}{j\pi\upsilon}$ then $\mathscr{F}\left(\frac{1}{\pi t}\right) = -j\,\text{sgn}(\upsilon)$

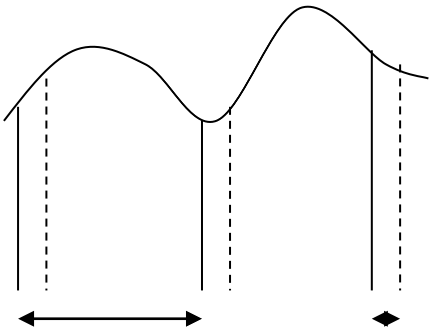
(The $\text{sgn}(.)$ function is odd)

We get : $\hat{\sigma}(\upsilon) = \mathscr{F}\left(H_i\left(s(t)\right)\right) = -j\,\text{sgn}(\upsilon).\hat{s}(\upsilon)$

1-That's mean that we get $\sigma(t)$ from $s(t)$ by a phase delay of $\pi/2$.



2- In the particular case of narrow band signal, around υ_0, a phase delay of $\pi/2$ means, to delay of a quarter of HF period. This delay can be done on the sampled signal. Also, we know that , for a narrow band signal $B\left(in\;\upsilon\rangle 0\right)$, a sample in HF is equivalent to two samples as shown in the graph.
The samples in continuous line correspond to s(t) and the samples in dot fed line, to $\sigma(t)$. The knowledge of the 2 series or samples is equivalent to the analytic signal, as we will see later.



3-Examples

If $s(t)=\cos\omega_0 t$ then $\sigma(t)=\sin\omega_0 t$
If $s(t)=\sin\omega_0 t$ then $\sigma(t)=-\cos\omega_0 t$
A few properties :

a) $s(t)\perp\sigma(t)$ because $\langle s(t),\sigma(t)\rangle=0$ (by Parseval's theorem)
b) $\sigma(t)=H_i\left[s(t)\right]$ *then* $s(t)=-H_i\left[\sigma(t)\right]$
c) $s(t)$ and $\sigma(t)$ have the same norm in L^2 $\|s(t)\|_2=\|\sigma(t)\|_2$
d) …

Analytic signal

By definition, the analytic signal associated to $s(t)$ is

$$\varphi(t)=s(t)+j\sigma(t)$$

Then of course :

$$\hat{\varphi}(\upsilon)=2H(\upsilon)\hat{s}(\upsilon) \quad \text{with} \quad H(\upsilon)=\text{Heaviside distribution}$$

Thus, $\varphi(t)$ is a signal with positive frequency components.
The hermician property of the Fourier transform is not conserved that means that it can't be a real signal.

Hilbert transform of $\sigma(t)$

We want to compute the Hilbert transform of $\sigma(t)$.

$$s(t)\xrightarrow{H_i}\sigma(t)\xrightarrow{H_i}\rho(t)$$

We have

$$\rho(t)=\frac{1}{\pi t}*\sigma(t)=\frac{1}{\pi t}*\frac{1}{\pi t}*s(t)$$

We can prove that

$$\frac{1}{\pi t}*\frac{1}{\pi t}=-\delta_{t=0}$$

Then :

$$H_i\left[H_i\left[s(t)\right]\right]=-s(t)$$

Remark :

$$\hat{\rho}(\upsilon)=\mathscr{F}\left[\rho(t)\right]=-i\,\mathrm{sgn}(\upsilon)\hat{\sigma}(\upsilon)=-i\,\mathrm{sgn}-i\,\mathrm{sgn}(\upsilon)\hat{s}(\upsilon)=-\hat{s}(\upsilon).$$

Bedrosian's theorem

If 2 functions $f(t)$ and $g(t)\in L^2$ with the Fourier transform $\hat{f}(\mu)$ and $\hat{g}(\mu)$ respectively with :

$$\hat{f}(\mu)=0 \quad for \quad |\mu|\rangle B$$
$$\hat{g}(\mu)=0 \quad for \quad |\mu|\langle B$$

Then :

$$H_i[f.g]=f.H_i[g]$$

Ceschi's theorem

If we consider 2 functions $f(t)$ and $\varphi(t)$ with :
1- $\varphi(t)$ is a complex rational function which can be put under the form :

$$\frac{p}{q} \quad with \quad d°q \geq d°p+1$$

q has only strictly negative imaginary part roots
2- $f(t)$ is a real rational function with the the positive or null imaginary part of the complex poles are equal to the zeros of $\varphi(t)$.
Then $\varphi(t)$ represents an analytic signal.
If writing $f\varphi$ under the form:

$$\frac{p_1}{q_1} \quad we \quad have \quad d°q_1 \geq d°p_1+1$$

Then $f\varphi$ represents also an analytic signal and noting g the real part of $\varphi(t)$

$$H_i[f.g]=f.H_i[g]$$

NB : we didn't use the hypothese of the Bedrosian theorem.

Causality

Writing that the transfer function of a stable system is the Fourier transform of a causal signal, i.e. of the impulse response $h(t)$ equal to zero if $t \leq 0$, we can write :

$$h(t)=H(t)h(t) \quad \text{H(t) : Heaviside distribution}$$

In the frequency domaine, we can write :

$$\hat{h}(\upsilon)=\frac{1}{2}\left[\delta_{\upsilon=0}+\frac{1}{j\pi\upsilon}\right]*\hat{h}(\upsilon)=-jH_i\left[\hat{h}(\upsilon)\right]$$

Writing now $\hat{h}(\upsilon)$ with its real and imaginary parts

$$\hat{h}(\upsilon) = a(\upsilon) + jb(\upsilon)$$
$$= -jH_i\left[a(\upsilon)\right] + H_i\left[b(\upsilon)\right]$$
$$\textit{then} \quad a(\upsilon) = H_i\left[b(\upsilon)\right]$$
$$\textit{and} \quad b(\upsilon) = -H_i\left[a(\upsilon)\right] \quad \textit{Kramers Kröenig} \quad \textit{relations}$$

Which means that the real and imaginary parts of the transfer function of a causal system are not independant. They are linked by the Hilbert's transform. If we know one of them, we know the other !

Example 1.

If $G = P + jQ$ $G = P + jQ \quad with \quad P(\omega) = \frac{1}{1+\omega^2}$, we can determine Q and G.

We can write : $Q(\omega) = -\frac{1}{\pi} VP \int_{-\infty}^{\infty} \frac{P(u)}{(\omega - u)} du = -\frac{1}{\pi} VP \int_{-\infty}^{\infty} \frac{1}{1+u^2} \frac{du}{(\omega - u)}$

And using the residual theorem we get :

$$Q(\omega) = \frac{-\omega}{1+\omega^2} \quad and \quad G = \frac{1}{1+\omega^2} - \frac{-j\omega}{1+\omega^2} = \frac{1}{1+j\omega}$$

Example 2.

The impedance of a capacity constitutes a transfer function $G(\omega) = Z(\omega) = \frac{-j}{C\omega}$. If we consider that the capacity is stable, the real part of $Z(\omega)$ can't be equal to zero, because $H_i(0) = 0 \neq \frac{-1}{C\omega}$. We want to determine this real part. Applying the last relation $a = H_i[b]$ we get with $B = \frac{-1}{C\omega}$;

$$a = \frac{1}{\pi\omega} * \frac{-1}{C\omega} = \frac{-\pi}{C}\left(\frac{1}{\pi\omega} * \frac{1}{\pi\omega}\right) = \frac{\pi}{C}\delta_{\omega=0}$$

Thus, it is more correct to write :

$$Z(\omega) = \frac{\pi}{C}\delta_{\omega=0} - \frac{j}{C\omega}$$

The impedance is infinite in $\omega = 0$ and complex

Another relationship between a and b.

As $\hat{h}(\upsilon)$ is hermitian, $a(\upsilon)$ is an even function and $b(\upsilon)$ an odd function. Let us explore the impulse response.

$$h(t) = \int_{-\infty}^{\infty} \hat{h}(\upsilon) e^{j2\pi\upsilon t} d\upsilon = \int_{-\infty}^{\infty} \left(a(\upsilon) + jb(\upsilon)\right) e^{j2\pi\upsilon t} d\upsilon$$
$$= 2\int_0^{\infty} a(\upsilon)\cos 2\pi\upsilon t.d\upsilon - 2\int_0^{\infty} b(\upsilon)\sin 2\pi\upsilon t.d\upsilon$$

Like $h(t) = 0 \quad if \quad t\langle 0$

We get :

$$\int_0^{\infty} a(\upsilon)\cos 2\pi\upsilon t.d\upsilon = \int_0^{\infty} b(\upsilon)\sin 2\pi\upsilon t.d\upsilon \qquad if \quad t\langle 0$$

Changing t by –t,

$$\int_0^{\infty} a(\upsilon)\cos 2\pi\upsilon t.d\upsilon = -\int_0^{\infty} b(\upsilon)\sin 2\pi\upsilon t.d\upsilon \qquad if \quad t\rangle 0$$

Thus we find the result :

$$h(t)=4\int_0^{\infty} a(\upsilon)\cos 2\pi\upsilon t.d\upsilon=-4\int_0^{\infty} b(\upsilon)\sin 2\pi\upsilon t.d\upsilon$$

Relation between gain and phase

When a transfer function $\hat{h}(\upsilon)$ has no pole in the right plane or on the Im ( ) axis and no zero in the right plane and Im ( ) axis too, then $\ln\hat{h}(\upsilon)$ is a function without singularity in the right plane and we can show that :

$$\ln\hat{h}(\upsilon)=\ln\left|\hat{h}(\upsilon)\right|+j.\arg\hat{h}(\upsilon)$$

has the same properties, that means that the gain and phase are not independant. The Hilbert relationship links them by :

$$\ln\left|\hat{h}(\upsilon)\right|=\frac{1}{\pi}VP\int_{-\infty}^{\infty}\frac{\arg\hat{h}(u)}{\upsilon-u}du$$

$$\arg\hat{h}(\upsilon)=\frac{-1}{\pi}VP\int_{-\infty}^{\infty}\frac{\ln\left|\hat{h}(u)\right|}{\upsilon-u}du$$

It is the case of the transfer function of minimum-phase,

Module and argument of the analytic function of s(t)

We have :

$$\varphi=s+j\sigma$$

And $\qquad |\varphi|=\sqrt{s^2+\sigma^2}$

Writing $\qquad \varphi\varphi^{'}=ss^{'}+\sigma\sigma^{'}\longrightarrow\begin{cases}|\varphi|\geq s\\ |\varphi|=|s|\Rightarrow\varphi^{'}=s^{'}\end{cases}$

Thus, the module of $\varphi(t)$ is always greater or equal to s(t) and has the same tan at the contact points. It is why, $\varphi(t)$ is called the complex envelope of s(t) (or of $\sigma(t)$).

Narrow band signal and analytic signal with carrier ω_0

If s(t) is a narrow band signal, we can write :

$$s(t)=e(t)\cos\left[\omega_0 t+\alpha(t)\right]$$

With $e(t)$ and $\alpha(t)$ having slow variations in front of $\frac{2\pi}{\omega_0}$. Let us seek $\sigma(t)$ the Hilbert transform of $s(t)$. In the frequency space, we get $\hat{\sigma}(\upsilon)$ by $\hat{\sigma}(\upsilon)=\mathscr{F}\left(H_i\left(s(t)\right)\right)=-j\,\mathrm{sgn}(\upsilon).\hat{s}(\upsilon)$

That means a phase delay of $\frac{-\pi}{2}$ for every frequency. But $\hat{s}(\upsilon)$ is a narrow band signal, this is equivalent to a delay of a quarter of period HF of $\hat{s}(\upsilon)$ equal to $\frac{1}{4}.\frac{2\pi}{\omega_0}$; then :

$$\sigma(t)\simeq e(t-\tau)\cos\left[\omega_0\left(t-\tau\right)+\alpha\left(t-\tau\right)\right]$$

$$=e(t-\tau)\cos\left[\left(\omega_0 t-\frac{\pi}{2}\right)+\alpha\left(t-\tau\right)\right]$$

But $e(t-\tau)\simeq e(t)\quad and\quad \alpha(t-\tau)\simeq\alpha(t)$ because we are with slow variations .

Thus

$$\sigma(t) \simeq e(t)\sin\left[\omega_0 t + \alpha(t)\right]$$

And :

$$\varphi(t) = s(t) + j.\sigma(t) \simeq e(t)\exp j\left(\omega_0 t + \alpha(t)\right) = \underbrace{\left[e(t)\exp j\alpha(t)\right]}_{complex\ envelope}\exp j\omega_0 t$$

Complex stochastic process

If s(t) is a complex stochastic process, so, $\sigma(t)$ is also a complex stochastic process. That means that s(t) and $\sigma(t)$ have the same spectral density power.

Let us compute the cross correlation function $B_{\sigma s}(\tau)$.

$$B_{\sigma s}\left(\tau\right) = E\left\{\sigma\left(t+\tau\right)s(t)\right\} = E\left\{\frac{1}{\pi}VP\int_{-\infty}^{\infty}\frac{s(u)}{t+\tau-u}du.s(t)\right\} = E\left\{\frac{1}{\pi}VP\int_{-\infty}^{\infty}\frac{s(u)s(t)}{t+\tau-u}du\right\}$$

$$= \frac{1}{\pi}VP\int_{-\infty}^{\infty}\frac{B_{ss}(u-t)}{\tau-(u-t)}du = \frac{1}{\pi}VP\int_{-\infty}^{\infty}\frac{B_{ss}(v)}{\tau-v}dv$$

Thus the cross correlation function of $\sigma(t)$ and s(t) is the Hilbert's transform of the correlation function of s(t).Also we can see that at the same time s(t) and $\sigma(t)$ are uncorrelated variables.

$$B_{\sigma s}\left(\tau\right) = \frac{1}{\pi}VP\int_{-\infty}^{\infty}\frac{B_{ss}(v)}{\tau-v}dv \quad and \quad B_{\sigma s}\left(0\right) = \frac{1}{\pi}VP\int_{-\infty}^{\infty}\frac{B_{ss}(v)}{-v}dv = 0$$

Because $B_{ss}(v)$ is even !

Application of the analytic signal to the SSB modulation.

We want to write the modulated signal in the SSB case by using the analytic signal.

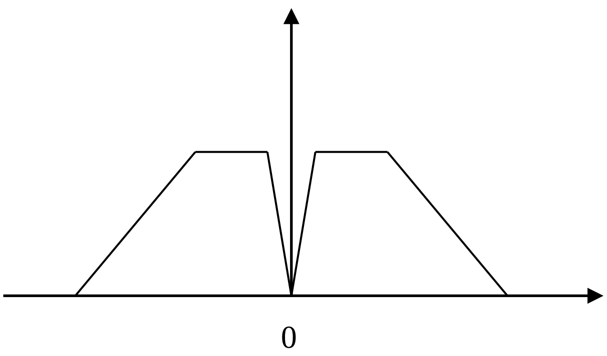
We call υ_0 the carrier, $x(t)$ the modulating signal and $\varphi(t)$ the analytic signal associated to $x(t)$.

Four steps.

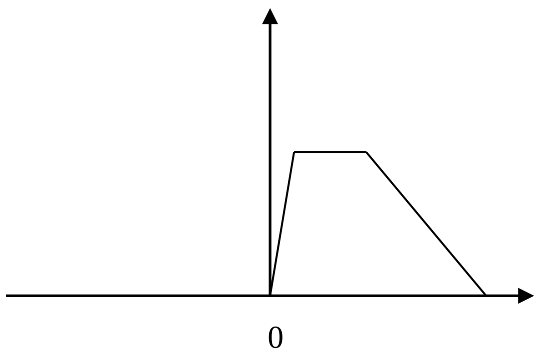
1-We build the analytic signal associated to $x(t)$.

$$\varphi(t) = x(t) + j\left(x(t) * \frac{1}{\pi t}\right)$$

Spectrum of $x(t)$



Spectrum of $\frac{1}{2}\varphi(t) = \hat{x}_+(\upsilon)$



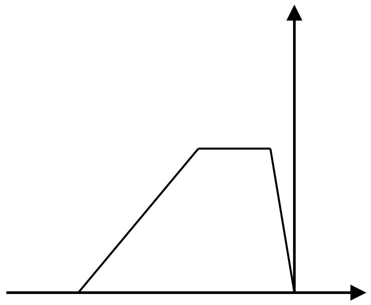
2-We translate the $\hat{x}_+(\upsilon)$ spectrum by υ_0 (equivalent operation by multiplying $e^{j2\pi\upsilon_0 t}$)

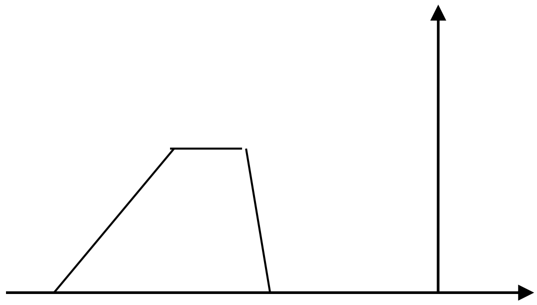
We get $\frac{1}{2}\varphi(t)e^{j2\pi\upsilon_0 t}$.

3-Do the same operation for the left side of the spectrum.

$$\psi(t)=x(t)-j\left(x(t)*\frac{1}{\pi t}\right)\xrightarrow{\mathscr{F}}\hat{x}(\upsilon)-j\left(-j\,\mathrm{sgn}(\upsilon)\,\hat{x}(\upsilon)\right)=2\hat{x}(\upsilon)H(-\upsilon)$$

Spectrum of $\frac{1}{2}\psi(t)=\hat{x}_-(\upsilon)$ Spectrum of $\frac{1}{2}\psi(t)e^{-j2\pi\upsilon_0 t}$





4-By adding the 2 spectrum we get $e(t)$ the emitted signal

$$e(t)=\frac{1}{2}\varphi(t)e^{j2\pi\upsilon_0 t}+\frac{1}{2}\psi(t)e^{-j2\pi\upsilon_0 t}$$

But we remark that $\varphi(t)$ *and* $\psi(t)$ are conjugated. Thus the emitted signal becomes :

$$e(t)=\mathcal{Re}\left[\varphi(t)e^{j2\pi\upsilon_0 t}\right]$$

Example :
If

$$x(t)=k\cos 2\pi\upsilon t$$
$$\varphi(t)=k\cos 2\pi\upsilon t+j.k\sin 2\pi\upsilon t=k.\exp 2\pi\upsilon t$$

The emitted $e(t)$ SSB is :

$$e(t)=\mathcal{Re}\left[\varphi(t)e^{j2\pi\upsilon_0 t}\right]=\mathcal{Re}\left[ke^{j2\pi\upsilon t}.e^{j2\pi\upsilon_0 t}\right]=\mathcal{Re}\left[ke^{j2\pi(\upsilon+\upsilon_0)t}\right]$$

Thus :

$$e(t)=k\cos 2\pi\left(\upsilon+\upsilon_0\right)t$$

Exercise:

We recall that $\sigma(t)$ is the Hilbert transform of $s(t)$ with the definition !

$$\sigma(t)=_{VP}\frac{1}{\pi t}*s\left(t\right)=_{VP}\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{s(x)}{t-x}dx$$

Give the Hilbert transform of :

a) $s\left(t-a\right)$; $a\in\mathbb{R}$ b) $\frac{d^2\left[s\left(t\right)\right]}{dt^2}$ c) $s(-t)$ d) $s(at)$ in fonction of $\sigma\left(\bullet\right)$.